ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation ▮ (▮▮▮▮) ▮▮▮–▮▮▮

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original articles

Monte Carlo method for solution of initial–boundary value problem for nonlinear parabolic equations

Abdujabar Rasulov[a,*], Gulnora Raimova[b]

[a] Mathematical Modeling and Informatics Department University of World Economy and Diplomacy, Tashkent, Uzbekistan
[b] Academy of Public Administration under the President of the Republic of Uzbekistan, Tashkent, Uzbekistan

Received 7 December 2013; received in revised form 6 April 2017; accepted 9 April 2017

Available online xxxx

Abstract

In this paper we will consider the initial–boundary value problem for a parabolic equation with a polynomial non-linearity relative to the unknown function. First we will derive a probabilistic representation of our problem. The representation of the solution of this problem is given in the form of a mathematical expectation, which is determined based on trajectories of branching processes. Under the assumption of the existence of the solution, an unbiased estimator is built using trajectories of a branching process. We will use a mean value theorem to write out a special integral equation, that equates the value of the unknown function $u(x,t)$ with its integral over a spheroid and balloid with center at the point (x,t). A probabilistic representation of the solution to the problem in the form of mathematical expectation of some random variables is obtained. This probabilistic representation uses a branching process whose trajectories are used in the contraction of an unbiased estimator for the solution. The derived unbiased estimator has a finite variance, and is built up from trajectories of branching processes with a finite average number of branches. Finally, the results of numerical experiments and application to the practical problems are discussed.

Keywords: Monte Carlo method; Branching random process; Martingale; Unbiased estimator

1. Introduction

In this paper we will consider the initial–boundary value problem for a parabolic equation with a polynomial non-linearity relative to the unknown function $u(x,t)$.

$$\frac{\partial u(x,t)}{\partial t} - a\Delta u(x,t) + cu(x,t) = f(u),$$

where $f(u) = \sum_{n=0}^{\infty} a_n(x,t)\, u^n(x,t)$. Theorems on the existence of solution for nonlinear equation are given in [4,8]. The method of statistical modeling for solving the initial–boundary value problem for a linear equation is considered

* Corresponding author.
E-mail addresses: asrasulov@gmail.com (A. Rasulov), raimova27@gmail.com (G. Raimova).

http://dx.doi.org/10.1016/j.matcom.2017.04.003

Please cite this article in press as: A. Rasulov, G. Raimova, Monte Carlo method for solution of initial–boundary value problem for nonlinear parabolic equations, Math. Comput. Simulation (2017), http://dx.doi.org/10.1016/j.matcom.2017.04.003

in [2] and also the case, when the right-hand side is finite series, was considered in [1,3,9,11]. The representation of the solution of this problem in the present work is given in the form of a mathematical expectation, which is determined based on trajectories of branching processes. Further, in this paper some results of the above mentioned works will be used to derive probabilistic representation for our problem. In the present paper under the assumption of the existence of the solution, an unbiased estimator is built using trajectories of a branching process. We will use a mean theorem value to write out a special integral equation, that equates the value of the function $u(x,t)$ with its integral over a spheroid and balloid with center at the point (x,t). A probabilistic representation of the solution to the problem in the form of mathematical expectation of some random variables is obtained. This probabilistic representation uses a branching process whose trajectories are used in the construction of an unbiased estimator for the solution. The derived unbiased estimator has a finite variance, and is built up from trajectories of branching processes with a finite average number of branches.

2. Description of the problem

Let D be a bounded domain in $\mathbb{R}^m$ with boundary ∂D and $\Omega = D \times [0,T]$ is a cylinder in $\mathbb{R}^{m+1}$. The following relations define functions $y_0(x) \in C\left(\bar{D}\right)$, $y(x,t) \in C\left(\partial D \times [0,T]\right)$, $f(x,t) \in C\left(\bar{\Omega}\right)$, $a_n(x,t) \in C\left(\bar{\Omega}\right)$ $(n = 0,1,2,\ldots)$ and the coefficients are $a > 0$, $c > 0$.

Let us consider the initial–boundary value problem for the following parabolic equation for $(x,t) \in \Omega$:

$$\frac{\partial u(x,t)}{\partial t} - a\Delta u(x,t) + cu(x,t) = \sum_{n=0}^{\infty} a_n(x,t)u^n(x,t), \tag{1}$$

with the initial and boundary conditions:

$$\begin{cases} u(x,t) = y(x,t), & x \in \partial D,\ t \in [0,T], \\ u(x,0) = y_0(x), & x \in D. \end{cases} \tag{2}$$

Assume functions $a_n(x,t)$, $y_0(x)$, $y(x,t)$ and coefficients a, c are such that there exists a unique continuous solution of this nonlinear problem $u(x,t) \in C\left(\bar{D} \times [0,T]\right) \cap C^{2,1}\left(D \times [0,T]\right)$ [4, p. 201], [8, p. 556]. Relative to the functions $a_n(x,t)$, $(n = 0,1,2,\ldots)$ we will make the following assumptions:

$$\sup_{(x,t)\in\Omega} |a_n(x,t)| \le \bar{a}_n, \quad \text{and the series } \sum_{k=0}^{\infty} \bar{a}_k k < \infty. \tag{3}$$

The purpose of this paper is to derive unbiased Monte Carlo estimators for solving the problem (1)–(2) with a finite variance at some arbitrary point $(x,t) \in \Omega$.

3. Integral representation of the solution

The basis for building the derived unbiased estimators will be the formula for the "parabolic mean" used in solving the heat equation. Various equations for the mean of parabolic equations were considered in [6,7]. With the help of the fundamental solution, and Green's formula, we can transform these differential equations into integral equations. In doing so we apply the results of Lemma 3.1 [12, p. 106]. We will obtain a special mean equation which equates the value of the function $u(x,t)$ with its integral over a balloid and its surface with center at the point (x,t).

As is known, the fundamental solution $Z(x,t;y,\tau)$ of the heat equation $u_t - a\Delta u = 0$ is

$$Z(x,t;y,\tau) = (4\pi\, a\ (t-\tau))^{-m/2} \exp\left(-\frac{|x-y|^2}{4\,a\,(t-\tau)}\right).$$

Let $Z_r(x,t;y,\tau) = Z(x,t;y,\tau) - (4\pi\, ar)^{-m/2}$. With the help of this fundamental solution, we define the family of domains $Q_r(x,t)$, which depend on the parameter $r > 0$ and the point $(x,t) \in \mathbb{R}^{m+1}$, as

$$Q_r(x,t) = \left\{(y,\tau) :\ Z(x,t;y,\tau) > (4\pi\, a\, r)^{-m/2}, \tau < t\right\}.$$