Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 145 (2018) 136–155
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Direct tensor-product solution of one-dimensional elliptic equations with parameter-dependent coefficients

Sergey V. Dolgov[a,*], Vladimir A. Kazeev[b], Boris N. Khoromskij[c]

[a] *University of Bath, Claverton Down, BA2 7AY Bath, United Kingdom*
[b] *Department of Mathematics, Stanford University, 450 Serra Mall, 94305 Stanford, USA*
[c] *Max-Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany*

Received 6 November 2014; received in revised form 10 October 2017; accepted 11 October 2017
Available online 27 October 2017

Abstract

We consider a one-dimensional second-order elliptic equation with a high-dimensional parameter in a hypercube as a parametric domain. Such a problem arises, for example, from the Karhunen–Loève expansion of a stochastic PDE posed in a one-dimensional physical domain. For the discretization in the parametric domain we use the collocation on a tensor-product grid. The paper is focused on the tensor-structured solution of the resulting multiparametric problem, which allows to avoid the curse of dimensionality owing to the use of the separation of parametric variables in the tensor train and quantized tensor train formats.

We suggest an efficient tensor-structured preconditioning of the entire multiparametric family of one-dimensional elliptic problems and arrive at a direct solution formula. We compare this method to a tensor-structured preconditioned GMRES solver in a series of numerical experiments.
© 2017 International Association for Mathematics and Computers in Simulation (IMACS). Published by Elsevier B.V. All rights reserved.

Keywords: Elliptic equations; Parametric problems; Tensor formats; Sherman–Morrison correction; Preconditioning

1. Introduction

We consider a multiparametric one-dimensional linear elliptic problem posed in the physical domain $D = (0, 1)$ and the parameter domain $\boldsymbol{Y} = [-1, 1]^M$:

$$\begin{aligned} -\frac{\partial}{\partial x} a(x, y) \frac{\partial u(x, y)}{\partial x} &= f(x, y), \quad x \in D, \quad y \in \boldsymbol{Y}, \\ u &= 0, \qquad\quad\; x \in \partial D, \; y \in \boldsymbol{Y}. \end{aligned} \tag{1.1}$$

Here, y denotes a tuple of parameters, $y = (y_1, \ldots, y_M) \in \boldsymbol{Y}$. For the solution of (1.1), we use the classical Finite Element Method (FEM) for the discretization in the physical domain D, and the collocation method [1] in

* Corresponding author.
E-mail addresses: S.Dolgov@bath.ac.uk (S.V. Dolgov), kazeev@stanford.edu (V.A. Kazeev), bokh@mis.mpg.de (B.N. Khoromskij).

https://doi.org/10.1016/j.matcom.2017.10.009
0378-4754/© 2017 International Association for Mathematics and Computers in Simulation (IMACS). Published by Elsevier B.V. All rights reserved.

the parameter domain $\boldsymbol{Y}$. More details are given in Section 3.1. For every $y \in \boldsymbol{Y}$ the problem (1.1) reduces to the form

$$\begin{aligned} -\frac{d}{dx}a(x)\frac{du(x)}{dx} &= f(x), \quad x \in D, \\ u &= 0, \qquad x \in \partial D. \end{aligned} \tag{1.2}$$

If we consider a standard FEM discretization of (1.2) with N degrees of freedom, it involves a tridiagonal $N \times N$-matrix, and the discrete problem is solved easily by the Gaussian elimination algorithm through $\mathcal{O}(N)$ operations.

Assume that we apply the same discretization to (1.1) and also discretize the problem with respect to the parameters by choosing a finite set of collocation points $y_s \in \boldsymbol{Y}$, $s = 1, \ldots, S$ and considering a set of problems (1.2) with $y = y_s$, $s = 1, \ldots, S$. This results in a large linear system with a block-diagonal matrix, which consists of S blocks, each of them being the tridiagonal matrix corresponding to a fixed parameter $y = y_s$. The direct methods cannot be applied to such a discretization straightforwardly for the solution of the whole family of problems, parameterized by a high-dimensional parameter $y \in \boldsymbol{Y}$. The reason for that is that the elementwise representation of the data itself suffers from the so-called “curse of dimensionality” with respect to M, which can be beyond tens or hundreds. For example, if the problem (1.1) approximates a stochastic PDE, the dimensionality M of the parametric space is governed by the accuracy of this approximation and, thus, may need to be high.

In the present paper we use the collocation on tensor-product uniform grids in $\boldsymbol{Y}$ with $S = n^M$ and employ the tensor-structured representation of the discrete problem to extract the “effective” degrees of freedom adaptively and make the problem tractable. A broad overview of this methodology can be seen in recent surveys [33,19,18,28]. This tensor approach was first applied to parametric and stochastic PDEs in [32] based on the *canonical* format and it was further extended in [34,31,15] to the case of *Hierarchical Tucker*, as well as *Tensor Train* [41] and *Quantized Tensor Train* formats. In this paper, we use the Quantized Tensor Train (QTT) format [38,27,39], which we describe in Section 2, and then show that the preconditioned system can be solved efficiently either by a direct method or by Krylov iterations demonstrating that these QTT-structured representations of the data appear to be highly efficient for the solution of the multiparametric problem.

We note that the algebraic construction using a direct solution formula in the present paper is motivated by the available results on low-rank approximate inverses of parametric elliptic operators [28] and on the structure of the reciprocal preconditioner [10]. It is due to the direct formula, available only in the case of one physical dimension, that the discretizations of parametric PDEs can be solved fully using only the basic algorithms of the TT-structured arithmetic. For higher-dimensional problems, efficient (but heuristic) tensor algorithms based on cross approximation are being developed [3,13].

The sparse grid approach [37,48,6,35], best N-term approximations [7] and adaptive Galerkin discretizations [14] could be used instead of tensor-structured representations to determine a reduced collocation set and, consequently, the complexity of the resulting problem. Further cost reduction in the sparse grid collocation or stochastic Galerkin methods can be achieved by a low-rank separation of physical and stochastic variables [15,36,5]. Another state-of-the-art technique for various high-dimensional problems, including stochastic PDEs, is the Monte Carlo and Quasi Monte Carlo methods [16,17,47].

We adhere to the collocation on full tensor-product grids and let the QTT format select the “effective” degrees of freedom adaptively. In the extreme case, when the *QTT ranks* are bounded by r uniformly in n and M, the QTT format ensures *logarithmic* complexity $\mathcal{O}\left(Mr^2 \log n\right)$. For parametric problems, although the TT ranks often exhibit the dependence $r \sim M$, the total storage $\mathcal{O}\left(M^3\right)$ is still much smaller than n^M. This makes the problem tractable for high M. The QTT representation may be used also for the physical variable $x \in D$.

The basics of the QTT-approximation theory for function-related tensors were given in [27,27]. Since then, the QTT format has been successfully applied to such problems as the solution of linear systems, eigenvalue problems, and in the discrete Fourier analysis, see [26,43,30,29,24,23,11,25]. There are also results on the analytical QTT structure of function-related vectors and discretized operators, see [21,42,22,44].

We propose two ways to solve (1.1) in the QTT representation. First, Krylov solvers can be preconditioned by the approach introduced in [10] for problems of the form (1.2). This preconditioner is based on the reciprocal diffusion coefficient $1/a$ and, in the one-dimensional case, ensures that the preconditioned matrix of the discrete problem has only two different eigenvalues. In the present paper we generalize this preconditioner to the multiparametric problem (1.1) and show how it can be used to precondition tensor-structured Krylov iterative solvers of (1.1). In particular, in our numerical experiments we consider the GMRES method.

Download English Version:

https://daneshyari.com/en/article/7543302

Download Persian Version:

https://daneshyari.com/article/7543302

Daneshyari.com