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Model order reduction and low-dimensional representations for random linear dynamical systems

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Abstract

We consider linear dynamical systems of ordinary differential equations or differential algebraic equations. Physical parameters are substituted by random variables for an uncertainty quantification. We expand the state variables as well as a quantity of interest into an orthogonal system of basis functions, which depend on the random variables. For example, polynomial chaos expansions are applicable. The stochastic Galerkin method yields a larger linear dynamical system, whose solution approximates the unknown coefficients in the expansions. The Hardy norms of the transfer function provide information about the input–output behaviour of the Galerkin system. We investigate two approaches to construct a low-dimensional representation of the quantity of interest, which can also be interpreted as a sparse representation. Firstly, a standard basis is reduced by the omission of basis functions, whose accompanying Hardy norms are relatively small. Secondly, a projection-based model order reduction is applied to the Galerkin system and allows for the definition of new basis functions within a low-dimensional representation. In both cases, we prove error bounds on the low-dimensional approximation with respect to Hardy norms. Numerical experiments are demonstrated for two test examples.

Keywords: Linear dynamical systems; Orthogonal expansion; Polynomial chaos; Model order reduction; Hardy norms

1. Introduction

In science and engineering, mathematical modelling often yields dynamical systems of ordinary differential equations (ODEs) or differential algebraic equations (DAEs). We focus on linear time-invariant dynamical systems. A quantity of interest is defined as an output of the problem. Physical parameters of the systems may exhibit uncertainties due to measurement errors or imperfections of an industrial manufacturing, for example. The uncertainties are described by the introduction of random variables. Since often many parameters appear in a system, we are interested in the case of high numbers of random variables.

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We expand the state variables as well as the quantity of interest into an orthonormal system of basis functions depending on the random variables. For example, the expansions of the polynomial chaos can be used, see [2,12,32,34]. Our aim is the construction of a low-dimensional approximation to the random quantity of interest, where only a few basis functions are required for a sufficiently accurate representation. Several previous works exist concerning the computation of such a low-dimensional approximation, which is sometimes called a sparse representation. As a tool was used, for example, least angle regression [7], sparse grid quadrature [8], compressed sensing [10] and ℓ_1-minimisation [18,19]. Our task can also be seen as an identification of a stochastic reduced basis, which was examined for random linear systems of algebraic equations in [22,28]. On the one hand, some methods start from a small set of basis functions and extend the basis successively until the approximation becomes sufficiently accurate. On the other hand, some techniques perform the choice of an initial set of basis functions, which is large and often yields a better accuracy than required, and reduce this basis. We apply strategies of the latter type.

Either a stochastic Galerkin method or a stochastic collocation technique yields approximations to the unknown coefficient functions in the expansions, see [23,24,32,33]. In this paper, we employ the stochastic Galerkin approach, which induces a larger linear dynamical system with many outputs. Hardy norms provide a measure for the importance of each output, where the $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm are used. Since the system becomes huge for large numbers of random parameters, a high potential for a model order reduction (MOR) appears. General theory on MOR can be found in the monographs [1,4,30], for example. We focus on projection-based techniques for the reduction of linear dynamical systems, see [13,14,16,17]. Projection-based MOR was applied to the stochastic Galerkin system in the previous works [21,25–27,35].

We investigate two strategies for the construction of a low-dimensional approximation. Firstly, a large initial basis is reduced by neglecting outputs of the Galerkin system with relatively small Hardy norms. This reduction implies directly a low-dimensional approximation to the random quantity of interest. Secondly, a general projection-based MOR technique decreases the dimensionality of the Galerkin system. We show that this MOR allows for the identification of a low-dimensional approximation to the random quantity of interest provided that the reduction achieves a sufficiently small system. In both cases, error bounds are proven for the low-dimensional representations with respect to Hardy norms.

The paper is organised as follows. In Section 2, we introduce the problem formulation and review already existing theory to be applied. The construction of a low-dimensional approximation by omitting basis functions is examined in Section 3. The definition of new basis functions using the information from an MOR is discussed in Section 4. We present numerical results for two illustrative examples in Section 5.

2. Problem definition

In this section, we define the problem under investigation. Furthermore, results from previous literature, which are relevant for our approaches, are outlined.

2.1. Linear dynamical systems

We consider a linear time-invariant system in descriptor form

$$\begin{aligned} E(p)\dot{x}(t,p) &= A(p)x(t,p) + B(p)u(t) \\ y(t,p) &= C(p)x(t,p), \end{aligned} \tag{1}$$

where the matrices $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_{\rm in}}$ and $C \in \mathbb{R}^{n_{\rm out}\times n}$ depend on physical parameters $p \in \Pi \subseteq \mathbb{R}^q$. The input $u : [0,\infty) \to \mathbb{R}^{n_{\rm in}}$ is supplied, while the output is defined by $y : [0,\infty)\times \Pi \to \mathbb{R}^{n_{\rm out}}$. If the matrix E is regular, then the system (1) consists of ODEs with state variables $x : [0,\infty)\times\Pi \to \mathbb{R}^n$. In our analysis, initial values $x(0,p) = 0$ are supposed for all $p \in \Pi$. If the matrix E is singular, then the system (1) represents DAEs with inner variables x. We restrict ourselves to the case of single-input–single-output (SISO) with $n_{\rm in} = n_{\rm out} = 1$, because generalisations to multiple-input–multiple-output (MIMO) are straightforward. We assume that the matrix pencil $\lambda E(p) - A(p)$ is regular for all $p \in \Pi$. Moreover, let the system (1) be stable for all $p \in \Pi$, i.e., the finite eigenvalues $\Sigma(p) \subset \mathbb{C}$ of the matrix pencil $\lambda E(p) - A(p)$ exhibit a negative real part.

The transfer function $H : (\mathbb{C} \setminus \Sigma(p)) \to \mathbb{C}$ characterises the input–output behaviour of the SISO system (1) in the frequency domain, see [1, Eq. (4.22)] for explicit ODEs or [14, Eq. (2.8)] for DAEs. This transfer function reads as

$$H(s,p) := C(p)(sE(p) - A(p))^{-1}B(p) \qquad \text{for } s \in \mathbb{C}\setminus\Sigma(p). \tag{2}$$