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Original Articles

Polynomial Hamiltonian systems of degree 3 with symmetric nilpotent centers

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Abstract

We provide normal forms and the global phase portraits in the Poincaré disk for all Hamiltonian planar polynomial vector fields of degree 3 symmetric with respect to the x-axis having a nilpotent center at the origin.

Keywords: Polynomial Hamiltonian systems; Nilpotent center; Phase portrait; Poincaré compactification

1. Introduction and statement of the results

Hamiltonian systems are relevant for many physical studies. Let $H(x, y)$ be a real polynomial in the variables x and y. Then a system of the form

$$x' = H_y(x, y) \qquad y' = -H_x(x, y) \tag{1}$$

is called a *polynomial Hamiltonian system*. Here the prime denotes derivative with respect to the independent variable t.

Poincaré in [20] defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point. Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that p is a center. Without loss of generality we can assume that p is at the origin of coordinates. Then after a linear change of variables and a rescaling of the time variable (if necessary), the

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system can be written in one of the following three forms

$$x' = -y + P(x, y), \qquad y' = x + Q(x, y), \tag{2}$$

$$x' = y + P(x, y), \qquad y' = Q(x, y), \tag{3}$$

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{4}$$

where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. In what follows a center of an analytic differential system in $\mathbb{R}^2$ is called *linear type*, *nilpotent* or *degenerate* if after an affine change of variables and a rescaling of the time it can be written as system (2), (3) or (4), respectively

The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic polynomial differential systems, and these results go back mainly to Dulac [12], Kapteyn [15,16] and Bautin [4]. In [21] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. For instance the linear type centers for cubic systems of the form linear plus homogeneous nonlinearities were characterized by Malkin [19], and by Vulpe and Sibirski [22]. For polynomial differential systems of the form linear plus homogeneous nonlinearities of degree greater than three the centers at the origin are not characterized, but there are partial results for degree four and five for the linear type centers, see for instance Chavarriga and Giné [5,6]. Some results for higher degree are known see for instance [14]. Recently Colak, Llibre and Valls [7–10] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. The complete classification of the phase portrait of the nilpotent centers in this last case was given in [11]. This has been possible since the classification of the nilpotent centers of system (2) when P and Q are homogeneous cubic polynomial was given in [1]. For a general overview on the centers of planar polynomial differential systems see [17], and for a classification of the phase portraits of same classes of other centers see [3] and the references quoted there. To know the phase portraits of centers is useful for studying the number of limit cycles which can bifurcate from their periodic orbits when they are perturbed, see for instance [18] and the references cited therein.

In this work we classify the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three symmetric with respect to the x-axis having a nilpotent center at the origin. We recall that the differential system (1) is *symmetric with respect to the x-axis* if it is invariant under the change of variables $(x, y, t) \to (x, -y, -t)$, sometimes this kind of systems are called *reversible*. The classification will be done using the Poincaré compactification of polynomial vector fields, see Section 2. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow.

Our main results are the following ones.

Proposition 1. *A Hamiltonian planar polynomial vector field of degree three with a nilpotent center at the origin and symmetric with respect to the x-axis, after a linear change of variables and a rescaling of its independent variable can be written as one of the following five classes:*

(I) $x' = y,\ y' = -x^3$;
(II) $x' = y + \delta y^3,\ y' = -x^3$;
(III) $x' = y + x^2 y + a y^3,\ y' = -x^3 - x y^2$;
(IV) $x' = y - x^2 y + a y^3,\ y' = -x^3 + x y^2$;
(V) $x' = y + 2xy + a x^2 y + b y^3,\ y' = -x^3 - y^2 - a x y^2$;

where $\delta \in \{-1, 1\}$ and $a, b \in \mathbb{R}$.

Proposition 1 is proved in Section 3.

Theorem 2. *The global phase portraits of the five families* (I)–(V) *in Proposition 1 are topologically equivalent to the phase portraits of Fig. 1:*

(a) 1.1 *for systems* (I), *systems* (II) *with $\delta = 1$, systems* (III) *with $a \geq 0$, systems* (IV) *with $a \geq 1$, systems* (V) *with $(a, b) \in \tilde{R}_4$, and systems* (V) *with $a \geq 1$ and $b = 0$;*
(b) 1.2 *for systems* (II) *with $\delta = -1$, and systems* (III) *with $a < 0$;*

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