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N-dimensional error control multiresolution algorithms for the cell average discretization

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Abstract

We present N-dimensional multiresolution algorithms with error control strategies in the cell average setting as a generalization to N dimensions of the work done in this direction. We present results proving the stability and giving explicit error bounds. We also explain how to carry out the programming and we include two numerical experiments to exemplify the utility of these algorithms.

Keywords: Multiresolution schemes; Cell averages; Error control; Nonlinearity

1. Introduction

Nowadays, multiresolution algorithms are useful tools in the field of signal processing, and in particular in image compression and denoising. And why have they become so popular? The answer is quite direct and easy: they proportion fast algorithms and good performances in comparison to other classical approaches such as Fourier based methods.

Given the data vector $f^L = (f_1^L, \dots, f_j^L, \dots)$ where L determines the resolution level, one can define a multiresolution representation of f^L as the sequence $\{f^0, d^1, \dots, d^L\}$ where f^k is an approximation of f^L at resolution $k < L$ and d^{k+1} represents the details required to recover f^{k+1} from f^k. The cardinality of the set $\{f^k, d^{k+1}\}$ is exactly the same as that of f^{k+1} and consequently the same thing occurs between $\{f^0, d^1, \dots, d^L\}$ and f^L. Moreover, it is straightforward to obtain the decoding algorithm for a given codification.

Inside the framework of multiresolution algorithms, we can differentiate between linear and nonlinear ones. Wavelet decompositions are key examples widely used for signal analysis [2,14] and in the solution of certain types of partial differential equations, since they lead to the solution of well-conditioned linear systems of equations and

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computationally fast algorithms. Wavelets nature however is inherently linear, which limits the performance attained with non-smooth input data. Several nonlinear alternatives such as ridgelets, bandelets and curvelets have appeared in the last years and since then they have been the matter of study of many scientists for different problems (see for example [21]).

Nonlinearity is naturally introduced in Harten's multiresolution algorithms [12,13]. In this framework, discrete resolution levels are connected by inter-resolution operators, named decimation (from fine (k) to coarse ($k-1$)) and prediction (from coarse to fine). These inter-scale operators are directly related to the *discretization* and *reconstruction* operators, which act between the continuous level (where a function f, related to the discrete data, lives) to each discrete level (where f^k lives). Discretization and decimation operators must be linear, but the reconstruction operator and in turn the prediction operator can be nonlinear and so better adapted to the given data. The greatest advantage of Harten's general framework lies precisely in its adaptability. The fundamental role played by the reconstruction operator makes it possible to perform *specific adaptive treatments at singularities*. In general, these involve data-dependent reconstruction operators leading to nonlinear prediction schemes and, hence, to nonlinear multiresolution decompositions [12,13,5].

Different settings can be considered depending on the linear discretization operator that produces the data. Classical settings are provided by the sampling operator (point value setting) or the averaging operators (spline settings). Linear multiresolution representations, such as wavelet decompositions, are related to data-independent reconstruction operators, and therefore linear prediction operators (see [8] for more details).

A similar framework for multiresolution was developed independently by Sweldens (see [17–19]).

A crucial issue in applying nonlinear multiresolution decompositions is the question of stability, in the following sense.

The multiresolution representation $\{f^0, d^1, \ldots, d^L\}$ of a signal is well adapted to data compression procedures. This multi-scale representation is then processed (truncation or quantization for instance) and the final result of this step is a modified multi-scale representation $\{\hat{f}^0, \hat{d}^1, \hat{d}^2, \ldots, \hat{d}^L\}$ which is *close* to the original one, i.e. such that (in some norm)

$$\|\hat{f}^0 - f^0\| \leq \epsilon_0 \qquad \|\hat{d}^k - d^k\| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \ldots, \epsilon_L$ are chosen according to some criteria specified by the user.

After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set f^L. In order for this to be true, some form of stability is needed, i.e. we must require that

$$\|\hat{f}^L - f^L\| \leq \sigma(\epsilon_0, \epsilon_1, \ldots, \epsilon_L), \tag{1}$$

where $\sigma(\cdot, \ldots, \cdot)$ satisfies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \ldots, \epsilon_L) = 0.$$

Linear multiresolution schemes are known to be stable, but this is not true in general for the nonlinear case. Many studies in this direction have been carried out in the last years by several authors. Let us mention the works of Amat, Donat, Matei, Cohen, Dyn, Meignen and Tadnor for example (see [6,7,9,11,15,16]). In whatever case the stability for nonlinear multiresolution algorithms is not always possible to prove, but it can always be attained by using the error control algorithms introduced by Harten [13]. In this sense it is also interesting to read the works of Sweldens [10]. These algorithms are based on a modified decoding algorithm, which keeps track of the total cumulative error, giving explicit error bounds dependent only on the truncation or quantization parameters.

The algorithms presented in this paper can be considered as much a generalization of Harten's error control algorithms in 1D as a particularization to the N-dimensional case in a rectangle of the completely general case studied by Harten [12,13]. The relevance of dealing with the details of each dimension are let clear in the 2D, and 3D cases in [1,2,4]. Although it does not add much more analytical complexity, it facilitates the programming and it clarifies the particular use to the potential user. Our purpose in this paper is both to generalize the algorithms and the theoretical results for whichever dimension obtaining concrete bounds for the decoding process in these versions of the error control algorithms. We exemplify their utility by presenting an experiment in 3D with a video sequence to carry out the approximation stage of a potential video compression algorithm and another experiment in 4D with a discontinuous function.