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Original Article

Numerical solution of parabolic Cauchy problems in planar corner domains

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Abstract

An iterative method for the parabolic Cauchy problem in planar domains having a finite number of corners is implemented based on boundary integral equations. At each iteration, mixed well-posed problems are solved for the same parabolic operator. The presence of corner points renders singularities of the solutions to these mixed problems, and this is handled with the use of weight functions together with, in the numerical implementation, mesh grading near the corners. The mixed problems are reformulated in terms of boundary integrals obtained via discretization of the time-derivative to obtain an elliptic system of partial differential equations. To numerically solve these integral equations a Nyström method with super-algebraic convergence order is employed. Numerical results are presented showing the feasibility of the proposed approach.

Keywords: Heat equation; Cauchy problem; Landweber method; Corner singularities; Boundary integral equation

1. Introduction

The Cauchy problem that we consider for parabolic equations consists of finding the solution which satisfies a given initial condition in a bounded planar domain D together with Cauchy boundary conditions (the function and its normal derivative) specified on a part of the boundary of the domain. This can model many different engineering applications, in particular, reconstruction of the temperature from incomplete boundary data. In the stationary case, there are numerous papers on solving the corresponding Cauchy problem via iterative methods based on the ideas presented in [16]. However, to the best of the authors' knowledge, for the time-dependent case, considerably fewer works based on such iterative methods have been presented, mainly [1,2,10,22]. The authors have recently investigated such iterative methods implemented via integral equations for the similar Cauchy problem but in doubly connected planar domains [6] and in semi-infinite planar regions [5]. The main limitation with these works is that the boundary part where Cauchy data is specified, has to be separated (positive distance) from the remaining part of the boundary

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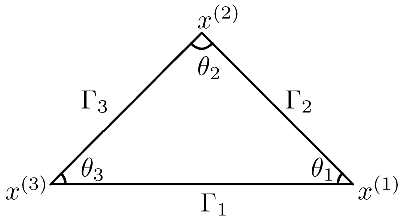

Fig. 1. Example of a corner domain D with $\Gamma_C = \Gamma_2 \cup \Gamma_3$, $\Gamma_I = \Gamma_1$.

of the given solution domain. In addition, the domains considered earlier had smooth boundary parts. In this study we make an attempt to overcome these limitations.

To formulate the problem to be studied, let $D \subset \mathbb{R}^2$ be a simply connected bounded domain, with boundary consisting of $L \geq 2$ open arcs $\Gamma_1, \ldots, \Gamma_L$, each being C^2-smooth. For ℓ=1, ..., L, the curves $\overline{\Gamma}_\ell$ and $\overline{\Gamma}_{\ell+1}$ are assumed to only have one point in common denoted by $x^{(\ell)}$ (here $\Gamma_{L+1} = \Gamma_1$). The interior angle θ_ℓ between these curves at the point $x^{(\ell)}$ is assumed to satisfy $0 < \theta_\ell < 2\pi$. Such a domain D is denoted a corner domain. Furthermore, we place every arc Γ_ℓ into one of two disjoint families: $\Gamma_\ell \in \Gamma_C$ – if Cauchy data are given on Γ_ℓ (C – "Cauchy"), otherwise $\Gamma_\ell \in \Gamma_I$ (I – "inaccessible"), see further Fig. 1 for an example of a solution domain D.

Let u (the temperature field) be a solution of the parabolic equation

$$\partial_t u - \Delta u = 0 \quad \text{in } D \times (0, T), \tag{1}$$

with, for simplicity, a homogeneous initial condition

$$u(x, 0) = 0 \quad \text{for } x \in D \tag{2}$$

and given boundary conditions (temperature and heat flux)

$$u = f_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on } \Gamma_C \times (0, T), \tag{3}$$

where $T > 0$ and f_1 is consistent with the initial condition. Note that no data is given on the inaccessible boundary part Γ_I. There is at most one solution to the Cauchy problem (1)–(3), see [23]. We assume that data are given such that there exists a solution. However, this solution does not in general depend continuously on the data.

To obtain a stable approximation, regularization methods are needed. Since the domain is non-smooth, we shall investigate ideas of a recent iterative method [13], where at each iteration step, mixed boundary problems are solved for (1) having Dirichlet data on Γ_I and Neumann data on Γ_C. This mixed problem is known as the Zaremba problem [25] and solutions to it can have singularities both at the corner points of the solution domain as well as at the points where the boundary condition changes type. Examples of parabolic problems and corner solutions domains where the solution can have singularities are given in [20,3]. Usually, in numerical methods for such problems subtraction of the singularities are used, see, for example, [19]. This can be a bit cumbersome sometimes to implement and knowledge of the asymptotic behaviour of the solution is needed. Therefore, we aim for a method where we do not need to subtract the singularity. Instead, the singularities are controlled via appropriate weights. Thus, we use weighted spaces and the method in [13] was analysed in weighted spaces of the type given in [15,24]. The weight is a certain power, involving a real number β, of the distance to the corner points of the solution domain. For the stationary case in planar corner domains a corresponding iterative method based on the boundary element method was investigated in [14] (for elliptic equations and corresponding weighted spaces see [15] and, for example, the monographs [17,21] and the references therein).

The novelty of this work is to then state an iterative Landweber-type procedure for solving the Cauchy problem (1)–(3) based on ideas in [12,13] for planar non-smooth domains, and to numerically implement this procedure, where there is no need to use the technique of subtracting singularities [3,19]. In particular, to formulate a suitable boundary integral formulation based on [7] where discretization of the time-derivative is used, carefully taking into account the singularities of the solution for the mixed problems in this procedure via weights and mesh grading. Moreover, to derive and present error estimates for the discretization of the direct problems using [4]. Note that the proposed approach makes use of knowledge of the initial data at $t = 0$. For the case when this data is not present, see [9].

The outline of this work is the following. In Section 2, we introduce some notation and spaces and give an iterative procedure (similar to [12] but for planar domains) for the Cauchy problem (1)–(3) and state some of its properties, for