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Upper bound on 3-rainbow domination in graphs with minimum degree 2

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ABSTRACT

Let $k \geq 1$ be an integer, and let G be a graph. A function $f : V(G) \to 2^{\{1,\dots,k\}}$ is a k-rainbow dominating function of G if every vertex $x \in V(G)$ with $f(x) = \emptyset$ satisfies $\bigcup_{y \in N_G(x)} f(y) = \{1, \dots, k\}$. The k-rainbow domination number of G, denoted by $\gamma_{rk}(G)$, is the minimum weight $w(f) = \sum_{x \in V(G)} |f(x)|$ of a k-rainbow dominating function f of G. In this paper, we prove that for every connected graph G of order $n \geq 8$ with $\delta(G) \geq 2$, $\gamma_{r3}(G) \leq \frac{5n}{6}$.

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex $x \in V(G)$, we let $N_G(x)$, $N_G[x]$ and $d_G(x)$ denote the *open neighborhood*, the *closed neighborhood* and the *degree* of x, respectively; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$. We let $\delta(G)$ and $\Delta(G)$ denote the *minimum degree* and the *maximum degree* of G, respectively. For vertices $x_1, x_2 \in V(G)$, we let $d_G(x_1, x_2)$ denote the *distance* between x_1 and x_2 in G. For a subset X of $V(G)$, we let $G[X]$ denote the subgraph of G induced by X. We let P_n and C_n denote the *path* and the *cycle* of order n, respectively. For a family $\mathcal{H}$ of subgraphs of G, we let $V(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} V(H)$. For terms and symbols not defined in this paper, we refer the reader to [1].

Let k be a positive integer, and let G be a graph. A function $f : V(G) \to 2^{\{1,\dots,k\}}$ is a *k-rainbow dominating function* (or a *k-RDF*) of G if every vertex $x \in V(G)$ with $f(x) = \emptyset$ satisfies $\bigcup_{y \in N_G(x)} f(y) = \{1, \dots, k\}$. For

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a function $f : V(G) \to 2^{\{1,\ldots,k\}}$, the value $w(f) := \sum_{x \in V(G)} |f(x)|$ is called the *weight* of f. The minimum weight of a k-RDF of G is called the *k-rainbow domination number* of G and is denoted by $\gamma_{rk}(G)$. A k-RDF of G having weight $\gamma_{rk}(G)$ is called a *γ_{rk}-function* of G. Note that the 1-rainbow domination number of G is equal to the domination number $\gamma(G)$ of G, which is a classical invariant in graph theory.

The concept of rainbow domination was defined by Brešar, Henning and Rall [2] in connection with a special guardman problem. Because of this, the definition of rainbow domination is often regarded as artificial. However, rainbow domination is also a mathematically meaningful concept. The 2-rainbow domination number of graphs has effectively been used in obtaining estimates of invariants concerning domination-like concepts such as domination, total domination and (weak) Roman domination (see [3–7]). Furthermore, k-rainbow domination type theorems related to Vizing's famous conjecture on domination were proved in [8]. Thus there is a hope that the study of rainbow domination will make a contribution to the solution of Vizing's conjecture. For such reasons, the rainbow domination number of graphs has widely been studied.

For a positive integer k and a graph G, the determination problem of the value $\gamma_{rk}(G)$ is NP-complete [9–11]. Thus to find sharp bounds of $\gamma_{rk}(G)$ is an important problem in the theory of rainbow domination. For example, the following results are known.

Theorem A (*Ore [12]*)**.** *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{r1}(G) \leq \frac{n}{2}$.*

Theorem B (*Wu and Rad [13]*)**.** *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{r2}(G) \leq \frac{3n}{4}$.*

Theorem C (*Fujita et al. [14]*)**.** *Let G be a connected graph of order $n \geq 5$. Then $\gamma_{r3}(G) \leq \frac{8n}{9}$.*

The above results are best possible, and Fink et al. [15] and Payan and Xuong [16] independently proved that every connected graph G ($\neq C_4$) of order n with $\gamma_{r1}(G) = \frac{n}{2}$ has endvertices. This suggests that if we make an additional assumption that $\delta(G) \geq 2$, then the bound in Theorem A can be improved. Indeed, the following result is well-known.

Theorem D (*McCuaig and Shepherd [17]*)**.** *Let G be a connected graph of order $n \geq 8$ with $\delta(G) \geq 2$. Then $\gamma_{r1}(G) \leq \frac{2n}{5}$.*

A sharp upper bound of the 2-rainbow domination number of graphs with minimum degree at least 2 is also known.

Theorem E (*Fujita and Furuya [5]*)**.** *Let G be a connected graph of order n with $\delta(G) \geq 2$. Then $\gamma_{r2}(G) \leq \frac{2n}{3}$.*

On the other hand, for an integer $k \geq 4$, we can easily verify that $\gamma_{rk}(C_n) = n$ (see [18]). This means that for $k \geq 4$, the trivial statement that $\gamma_{rk}(G) \leq n$ for any (connected) graph G of order n (with $\delta(G) \geq 2$) gives the best possible upper bound. In view of the results mentioned so far, we are naturally led to the problem of obtaining an upper bound of $\gamma_{r3}(G)$ for a connected graph G with $\delta(G) \geq 2$ in terms of its order. Our main purpose in this paper is to give such a bound as follows:

Theorem 1.1. *Let G be a connected graph of order $n \geq 8$ with $\delta(G) \geq 2$. Then $\gamma_{r3}(G) \leq \frac{5n}{6}$.*

The bound in Theorem 1.1 is best possible (see Lemma 4.3(i) in Section 4.2). Indeed, we prove a result stronger than Theorem 1.1. To state our main result, we need some definitions and notations.

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