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Staircase compatibility and its applications in scheduling and piecewise linearization

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ABSTRACT

We introduce the *Clique Problem with Multiple-Choice constraints (CPMC)* and characterize a case where it is possible to give an efficient description of the convex hull of its feasible solutions. This special case, which we name *staircase compatibility*, generalizes common properties in several applications and allows for a linear description of the integer feasible solutions to (CPMC) with a totally unimodular constraint matrix of polynomial size. We derive two such totally unimodular reformulations for the problem: one that is obtained by a strengthening of the compatibility constraints and one that is based on a representation as a dual network flow problem. Furthermore, we show a natural way to derive integral solutions from fractional solutions to the problem by determining integral extreme points generating this fractional solution. We also evaluate our reformulations from a computational point of view by applying them to two different real-world problem settings. The first one is a problem in railway timetabling, where we try to adapt a given timetable slightly such that energy costs from operating the trains are reduced. The second one is the piecewise linearization of non-linear network flow problems, illustrated at the example of gas networks. In both cases, we are able to reduce the solution times significantly by passing to the theoretically stronger formulations of the problem.

1. Introduction

Compatibility structures are prevalent in many combinatorial optimization problems. In fact, they arise whenever the choice of one solution element implies the choice of other elements — in the sense of a *compatibility constraint* of the form

$$x \leq \sum_{i \in C} y_i, \tag{1}$$

with binary variables x and y_i for $C \subseteq S := \{1, \ldots, n\}$.

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In this work, we consider the combination of compatibility constraints with another frequently-occurring structure, so-called *multiple-choice constraints* of the form

$$\sum_{i \in S} y_i = 1. \quad (2)$$

These constraints are present whenever there is a partition of the set of eligible elements into subsets, such that it is required to choose exactly one element from each subset.

Both types of constraints together imply a clique-type problem, which can be seen by subtracting (2) from (1), resulting in a stable-set constraint on a certain graph:

$$x + \sum_{i \in S \setminus C} y_i \leq 1.$$

Generalizing this concept to an arbitrary number of compatibility and multiple-choice relations between elements leads to a very interesting problem, which we name the *Clique Problem with Multiple-Choice constraints (CPMC)*: let $G = (V, E)$ be an undirected graph and $\mathcal{V} = \{V_1, \dots, V_m\}$ a partition of its node set V into m disjoint subsets such that each subset V_i is a stable set in G. The clique problem with multiple-choice constraints then asks to find a clique in G that contains exactly one node from each subset V_i.

While this problem is NP-hard in general (see [1] or [2]), we want to investigate here a special case where it is solvable in polynomial time. This is possible for a restriction of the compatibility graphs to graphs with a certain compatibility structure which we name *staircase compatibility*. For clique problems exhibiting this special structure, we will be able to state efficient linear programming (LP) formulations of which we can show that the corresponding constraint matrix is totally unimodular.

In order to demonstrate that there is great benefit from studying this structure, we present two very distinct real-world applications which can be modelled as (CPMC) under staircase compatibility. The first one is a problem in railway timetabling which falls into the class of project scheduling problems. We will see that the precedence constraints of the project scheduling problem have a very natural correspondence to the compatibility constraints in (CPMC). The second application arises in the context of piecewise-linear approximation of non-linear functions in optimization problems on transport networks. In both cases, the resulting model reformulations are already known (see [3] and [4] respectively). However, our notion of staircase compatibility provides a common, more general framework to study the underlying clique problem with multiple-choice constraints. In particular, we are able to show that the derived integrality results hold for a wider class of compatibility graphs.

We begin with the definition of staircase compatibility in Section 2, which is accompanied by a first discussion of its presence in project scheduling problems and flow problems with a piecewise-linear objective function as well as similar structures in the literature. Following that, we derive two totally unimodular LP formulations for the clique problem with multiple-choice constraints in the case of staircase compatibility in Section 3. The second of these two reformulations takes the form of a dual network flow problem. It will give rise to a very natural way of generating heuristic solutions from a fractional solution to the problem by determining integral extreme points which generate this fractional solution. In Section 4, we present our computational results for the two practical applications mentioned above, showing that the better understanding of their staircase structure directly translates into vastly shorter solution times. Finally, in Section 5, we summarize the findings of this paper and give possible directions for an extension of our results.

2. Staircase compatibility

In the following, we define the clique problem with multiple-choice constraints as it is considered here as well as the notion of staircase compatibility. We will see later that the presence of such staircase structures in the problem allows for an efficient description of the convex hull of its feasible solutions.