ARTICLE IN PRESS

Discrete Optimization ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Discrete Optimization

www.elsevier.com/locate/disopt

Complete formulations of polytopes related to extensions of assignment matrices

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ARTICLE INFO

Article history:
Received 4 September 2017
Received in revised form 22 February 2018
Accepted 13 April 2018
Available online xxxx

Keywords:
Assignment problems
Polytope
Separation
Complete formulation
Compact formulation

ABSTRACT

Let k, n denote two positive integers and consider the family of the polytopes defined as the convex hull of pairs of the form (Y, h) where Y is a 0/1-matrix with k rows, n columns, containing exactly one nonzero coefficient per column, and where h stands for the smallest index of a nonzero row of Y.

These polytopes and some variants naturally emerge in formulations of different classical combinatorial optimization problems such as minimum makespan scheduling and minimum span frequency assignment.

In this paper, we provide complete formulations for these polytopes and show the associated separation problem can be solved in polynomial time. The complete formulations in the original space of variables generally contain an exponential number of inequalities. Alternative extended compact formulations are also presented.

1. Introduction

Given a set S and two positive integers k, n, let $M_{k\times n}(S)$ denote the set of matrices having k rows, n columns and all of their entries in the set S. For a matrix $Y \in M_{k\times n}(S)$, let y_i^j denote its entry in the ith row and jth column. Given a set $S \subset \mathbb{R}^n$, $conv(S)$ stands for the convex hull of S. $\mathbb{N}^*$ stands for the set of positive integers. For $(q, n) \in \mathbb{N} \times \mathbb{N}^*$ with $q < n$, $[\![q, n]\!]$ stands for the set of integers from q to n: $[\![q, n]\!] = \{q, q+1, \ldots, n\}$, and for the case when $q = 1$ we also use $[n]$ to denote $[\![1, n]\!]$.

In this paper, we study the family of polytopes $(\mathcal{P}^{\min}_{k,n})_{(k,n)\in(\mathbb{N}^*)^2}$ defined as the convex hull of vectors of the form $(Y, h) \in M_{k\times n}(\{0, 1\}) \times \mathbb{N}^*$ such that each column of the matrix Y has exactly one nonzero entry, and the value of h corresponds to the smallest index of a row having at least one nonzero entry. Formally,

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https://doi.org/10.1016/j.disopt.2018.04.002
1572-5286/

$$\mathcal{P}^{\min}_{k,n} = conv\left(\left\{ (Y,h) \in M_{k\times n}(\{0,1\}) \times \mathbb{N}^* \;\middle|\; \begin{array}{l} \sum_{l=1}^{k} y^i_l = 1,\ \forall i \in [n], \\ h = \min_{i\in[n]} \sum_{l=1}^{k} l y^i_l \end{array} \right\}\right).$$

We define in the same manner the family of polytopes $(\mathcal{P}^{\max}_{k,n})_{(k,n)\in(\mathbb{N}^*)^2}$ replacing "min" with "max" in the definition above. Polytopes such as $\mathcal{P}^{\min}_{k,n}$ or $\mathcal{P}^{\max}_{k,n}$ naturally arise in combinatorial optimization problems involving n variables $(z^i)_{i=1}^n$ each of which can be assigned an integer number in $[k]$. Then each variable y^j_i has the following interpretation: $y^j_i = 1$ if and only if $z^j = i$. And for the case of $\mathcal{P}^{\min}_{k,n}$ (resp. $\mathcal{P}^{\max}_{k,n}$), the value of the variable h is the smallest (resp. largest) value assigned to the variables $(z^i)_{i=1}^n$. Next, we mention some related works which motivated our investigations, namely combinatorial optimization problems where these polytopes naturally emerge.

Motivation and related work

Given a set J of n jobs, a set M of m machines that can all process at most one job at a time, and the time $t^{i,j}$ for processing job $j \in J$ on machine $i \in M$, the goal of the *minimum makespan scheduling problem* is to assign a machine $i \in M$ for each job $j \in J$ so as to minimize the *makespan*, i.e. the maximum processing time of any machine (see e.g., [1]). Several approximation schemes have been developed to deal with this $\mathcal{NP}$-hard problem [2], e.g. [3] and [4]. The timeline is discretized into units of time (e.g., days) and the processing times are integers. Consider the variant where all the machines are identical [5] and the interruption of a task being processed is not allowed. So, in this case, for any job $j \in J$, $t^{i,j} = t^j,\ \forall i \in M$ and assigning a machine to each job reduces to assigning a day to be the last day of processing this job, which also determines the first day of the processing. We can then formulate the problem as follows. We take $k = \sum_{j=1}^n t^j$ and the variable $X \in M_{k\times n}(\{0,1\})$ whose interpretation is: $x^j_l = 1$ if and only if the processing of the job j ends on the day l. We obtain the following formulation.

$$\begin{cases} \min g \\ \quad \text{s.t. } \sum_{l=1}^{k} l x^j_l \geq t^j,\ \forall j \in [n], \\ \quad \sum_{j=1}^{n} \sum_{l'=l}^{\min(l+t^j-1,k)} x^j_{l'} \leq m,\ \forall l \in [k], \\ \quad (X,g) \in \mathcal{P}^{\max}_{k,n},\ X \in M_{k\times n}(\{0,1\}). \end{cases}$$

The first constraints ensure that for each job $j \in [n]$ its processing ends after enough time, while the second set of constraints ensure that no more than m jobs are processed daily. This formulation may be altered to consider additional constraints (such as precedence or release time) and any linear objective function.

The *minimum-span frequency-assignment problem* is a variant of the $\mathcal{NP}$-hard *frequency-assignment problem* [6]. The input is a graph $G = (V,E)$ called the *interference graph*, with $V = \{v_1,\ldots,v_n\}$ and $ij \in E$ iff the signals at the nodes (representing antennas) i and j can interfere. A frequency f from a set F of available frequencies (integer values) must be assigned to each node $v \in V$, in such a way that, for each edge $e \in E$, both endpoints are assigned with different frequencies. In addition, to reduce interferences, stronger requirements may be imposed: $|f(u) - f(v)| \geq s_{uv}, \forall uv \in E$, where $f(u)$ (resp. $f(v)$) stands for the frequency assigned to u (resp. v), and s_{uv} is a given threshold value. The problem consists of assigning frequencies to nodes taking into account the separation requirements and such that the difference between the maximum and minimum assigned numbers is minimized (see, e.g., [7]). With $F = [k]$ where k is an upper bound on the minimum span and using boolean variables x^i_l taking value 1 iff frequency l is assigned to node v_i, the problem can be formulated as follows.

$$\begin{cases} \min\ g \\ \quad \text{s.t. } x^i_l + x^j_{l'} \leq 1,\ \forall (i,j,l,l') \in [n]^2 \times [k]^2 \text{ such that } v_iv_j \in E, |l-l'| < s_{v_iv_j}, \\ \quad (X,g) \in \mathcal{P}^{\max}_{k,n},\ X \in M_{k\times n}(\{0,1\}). \end{cases}$$