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Total dominating sequences in trees, split graphs, and under modular decomposition

Boštjan Brešar[a,b,*], Tim Kos[b], Graciela Nasini[c], Pablo Torres[c]

[a] *Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia*
[b] *Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[c] *Depto. de Matemática, Universidad Nacional de Rosario and Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina*

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ABSTRACT

A sequence of vertices in a graph G with no isolated vertices is called a total dominating sequence if every vertex in the sequence totally dominates at least one vertex that was not totally dominated by preceding vertices in the sequence, and, at the end all vertices of G are totally dominated (by definition a vertex totally dominates its neighbors). The maximum length of a total dominating sequence is called the Grundy total domination number, $\gamma_{\rm gr}^t(G)$, of G, as introduced in Brešar et al. (2016). In this paper we continue the investigation of this concept, mainly from the algorithmic point of view. While it was known that the decision version of the problem is NP-complete in bipartite graphs, we show that this is also true if we restrict to split graphs. A linear time algorithm for determining the Grundy total domination number of an arbitrary forest T is presented, based on the formula $\gamma_{\rm gr}^t(T) = 2\tau(T)$, where $\tau(T)$ is the vertex cover number of T. A similar efficient algorithm is presented for bipartite distance-hereditary graphs. Using the modular decomposition of a graph, we present a frame for obtaining polynomial algorithms for this problem in classes of graphs having relatively simple modular subgraphs. In particular, a linear algorithm for determining the Grundy total domination number of P_4-tidy graphs is presented. In addition, we prove a realization result by exhibiting a family of graphs G_k such that $\gamma_{\rm gr}^t(G_k) = k$, for any $k \in \mathbb{Z}^+ \setminus \{1,3\}$, and showing that there are no graphs G with $\gamma_{\rm gr}^t(G) \in \{1,3\}$. We also present such a family, which has minimum possible order and size among all graphs with Grundy total domination number equal to k.

1. Introduction

The *total domination number*, $\gamma_t(G)$, of a graph G with no isolated vertices is the smallest cardinality of a set of vertices S such that every vertex of G has a neighbor in S. (If the condition only requires that vertices from $V(G) \setminus S$ have a neighbor in S, then the resulting invariant is the *domination number* $\gamma(G)$ of

* Corresponding author at: Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia.
E-mail address: bostjan.bresar@um.si (B. Brešar).

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G.) Let us introduce our main invariant, which is defined for all graphs G without isolated vertices, see [1]. Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. The sequence S is a *legal (open neighborhood) sequence* if

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{1}$$

holds for every $i \in \{2, \ldots, k\}$. If, in addition, $\widehat{S}$ is a total dominating set of G, then we call S a *total dominating sequence* of G. (Recall that a vertex u *totally dominates* a vertex v in a graph G if u and v are adjacent. A set S of vertices in a graph G is a *total dominating set* of G if every vertex of G is totally dominated by some vertex of S.) The maximum length of a total dominating sequence in G is called the *Grundy total domination number* of G and denoted by $\gamma_{\rm gr}^t(G)$; the corresponding sequence is called a *Grundy total dominating sequence* of G.

A motivation for introducing total dominating sequences came from the so-called total domination game [2,3], in which the sequences are a result of two-player game with players having the opposite goals; one player wants the graph to be totally dominated in as few moves as possible, while the other player wants to maximize the sequence of moves. The length of the resulting sequence played in such a game is thus a lower bound for the Grundy total domination number of a graph. A similar game, called the domination game, was introduced earlier with respect to the standard domination number [4], and was already studied in a number of papers. In particular, motivated by the domination game, a different version of dominating sequences was defined in [5], in which legality is considered with respect to closed neighborhoods (i.e., in the above definition just replace open neighborhoods by closed neighborhoods in (1)); longest sequences in that sense are called the *Grundy dominating sequences*, and the corresponding invariant the *Grundy domination number of a graph*.

Efficient algorithms for the Grundy domination number of trees, cographs and split graphs have been presented in [5]. In addition, minimal dominating sets have been characterized through some algebraic properties of dominating sequences, and some general lower bounds for this parameter were also established. Similarly, a lower bound was obtained for the Grundy total domination number of an arbitrary graph in [1], with an improvement for k-regular graphs. Using the connection with covering sequences in hypergraphs, NP-completeness of the decision version of the Grundy total domination number in bipartite graphs was also established. Nevertheless, no other algorithmic issues were considered in [1]. In this paper we continue the study of the Grundy total domination number with an emphasis on algorithmic issues. In particular, we will focus on a class of graphs that widely generalizes the class of cographs, and are called P_4-tidy graphs. This graph class generalizes several other graph classes having few P_4's, such as P_4-sparse, P_4-extendible and P_4-reducible graphs [6], see also [7]. These graphs are also high in the hierarchy of the classes of graphs recognizable by modular decomposition (cf. [8] for modular decomposition and [6] for representation of modular graphs with respect to P_4-tidy graphs). Therefore, by obtaining the Grundy total domination number with respect to modular decomposition and in P_4-tidy graphs, this resolves the Grundy domination problem for a number of other well-known graph classes.

In Section 3 we start by setting new bounds for the Grundy total domination number and proving a realization theorem about it. We continue in Section 4, in which we first note, how the two operations on which modular decomposition of a graph depends, namely the join and the disjoint union of two graphs, effect the Grundy total domination number. We follow with an application of these observations in Section 5, by presenting a linear algorithm to determine the Grundy total domination number of P_4-tidy graphs. Besides, we note a first difference between the (standard) Grundy domination number and the Grundy total domination number in their behavior in the class of trees. For the former no explicit formula was found [5], and the algorithm for determining the Grundy domination number of a tree is based on a recursive, dynamic