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Discrete convexity in joint winner property

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ABSTRACT

In this paper, we reveal a relation between joint winner property (JWP) in the field of valued constraint satisfaction problems (VCSPs) and M$^\natural$-convexity in the field of discrete convex analysis (DCA). We introduce the M$^\natural$-convex completion problem, and show that a function f satisfying the JWP is Z-free if and only if a certain function $\overline{f}$ associated with f is M$^\natural$-convex completable. This means that if a function is Z-free, then the function can be minimized in polynomial time via M$^\natural$-convex intersection algorithms. Furthermore we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

1. Introduction

A *valued constraint satisfaction problem (VCSP)* is a general framework for discrete optimization (see [1] for details). Informally, the VCSP framework deals with the minimization problem of a function represented as the sum of "small" arity functions. It is known that various kinds of combinatorial optimization problems can be formulated in the VCSP framework. In general, the VCSP is NP-hard. An important line of research is to investigate which classes of instances are solvable in polynomial time, and why these classes ensure polynomial time solvability. Cooper–Živný [2] showed that if a function represented as the sum of unary or binary functions satisfies the *joint winner property (JWP)*, then the function can be minimized in polynomial time. This gives an example of a class of instances that are solvable in polynomial time.

In this paper, we present the reason why JWP ensures polynomial time solvability via *discrete convex analysis (DCA)* [3], particularly, M$^\natural$-*convexity* [4]. DCA is a theory of convex functions on discrete structures, and M$^\natural$-convexity is one of the important convexity concepts in DCA. M$^\natural$-convexity appears in many areas such as operations research, economics, and game theory (see e.g., [3,5,6]).

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The results of this paper are summarized as follows:

- We reveal a relation between JWP and M$^\natural$-convexity. That is, we give a DCA interpretation of polynomial-time solvability of JWP.
- To describe the connection of JWP and M$^\natural$-convexity, we introduce the M$^\natural$-*convex completion problem*, and give a characterization of M$^\natural$-convex completability.
- By utilizing a DCA interpretation of JWP, we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

This study will hopefully be the first step towards fruitful interactions between VCSPs and DCA.

Notations. Let $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of reals and nonnegative reals, respectively. In this paper, functions can take the infinite value $+\infty$, where $a < +\infty$, $a + \infty = +\infty$ for $a \in \mathbf{R}$, and $0 \cdot (+\infty) = 0$. Let $\overline{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ and $\overline{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{+\infty\}$. For a function $f : \{0,1\}^n \to \overline{\mathbf{R}}$, the effective domain is denoted as $\mathrm{dom}\, f := \{x \in \{0,1\}^n \mid f(x) < +\infty\}$. For a positive integer k, we define $[k] := \{1, 2, \ldots, k\}$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, we define $\mathrm{supp}^+(x) := \{i \in [n] \mid x_i > 0\}$.

2. Preliminaries

Joint winner property. Let $d_i \geq 2$ be a positive integer and $D_i := [d_i]$ for $i \in [r]$. We consider a function $f : D_1 \times D_2 \times \cdots \times D_r \to \overline{\mathbf{R}}_+$ represented as the sum of unary or binary functions as

$$f(x_1, x_2, \ldots, x_r) = \sum_{i \in [r]} c_i(x_i) + \sum_{1 \leq i < j \leq r} c_{ij}(x_i, x_j), \tag{1}$$

where $c_i : D_i \to \mathbf{R}_+$ is a unary function for $i \in [r]$ and $c_{ij} : D_i \times D_j \to \overline{\mathbf{R}}_+$ is a binary function for $1 \leq i < j \leq r$. Furthermore we assume $c_{ij} = c_{ji}$ for distinct $i, j \in [r]$. A function f of the form (1) is said [2] to satisfy the *joint winner property (JWP)* if it holds that

$$c_{ij}(a, b) \geq \min\{c_{jk}(b, c), c_{ik}(a, c)\} \tag{2}$$

for all distinct $i, j, k \in [r]$ and all $a \in D_i, b \in D_j, c \in D_k$. A function f of the form (1) satisfying the JWP is said to be *Z-free* if it satisfies that

$$|\mathrm{argmin}\{c_{ij}(a, c), c_{ij}(a, d), c_{ij}(b, c), c_{ij}(b, d)\}| \geq 2 \tag{3}$$

for any $i, j \in [r]$ $(i \neq j)$, $\{a, b\} \subseteq D_i$ $(a \neq b)$, and $\{c, d\} \subseteq D_j$ $(c \neq d)$.

Cooper–Živný [2] showed that if f of the form (1) satisfies the JWP, then f can be minimized in polynomial time. In fact, they showed that if f satisfies the JWP, then f can be transformed into a certain Z-free function f' in polynomial time such that a minimizer of f' is also a minimizer of f. Moreover they showed that a Z-free function can be minimized in polynomial time.

JWP appears in many contexts. For example, JWP identifies a tractable class of the MAX-2SAT problem, which is a well-known NP-hard problem [7]. Indeed, for a 2-CNF formula ψ, we can represent the MAX-2SAT problem for ψ as a Boolean binary $\{0,1\}$-valued VCSP instance. In this binary VCSP instance, JWP is equivalent to the following condition on ψ: if clauses $(x_1 \vee x_2)$ and $(x_1 \vee x_3)$ are contained in ψ, then so is $(x_2 \vee x_3)$. In addition, the AllDifferent constraint [8] and SoftAllDiff constraint [9] can be regarded as special cases of the JWP, and certain scheduling problems introduced in [10,11] satisfy the JWP. See also Examples 5–8 in [2] for details.

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