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Elementary polytopes with high lift-and-project ranks for strong positive semidefinite operators☆

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ABSTRACT

We consider operators acting on convex subsets of the unit hypercube. These operators are used in constructing convex relaxations of combinatorial optimization problems presented as a 0,1 integer programming problem or a 0,1 polynomial optimization problem. Our focus is mostly on operators that, when expressed as a lift-and-project operator, involve the use of semidefiniteness constraints in the lifted space, including operators due to Lasserre and variants of the Sherali–Adams and Bienstock–Zuckerberg operators. We study the performance of these semidefinite-optimization-based lift-and-project operators on some elementary polytopes — hypercubes that are chipped (at least one vertex of the hypercube removed by intersection with a closed halfspace) or cropped (all 2^n vertices of the hypercube removed by intersection with 2^n closed halfspaces) to varying degrees of severity ρ. We prove bounds on ρ where the Sherali–Adams operator (strengthened by positive semidefiniteness) and the Lasserre operator require n iterations to compute the integer hull of the aforementioned examples, as well as instances where the Bienstock–Zuckerberg operators require $\Omega(\sqrt{n})$ iterations to return the integer hull of the chipped hypercube. We also show that the integrality gap of the chipped hypercube is invariant under the application of several lift-and-project operators of varying strengths.

1. Introduction

A foundational tool in tackling many combinatorial optimization problems is the construction of convex relaxations. Starting with a 0,1 integer programming formulation of the given problem, the goal is to find a tractable (whether in practice or theory, hopefully in both) optimization problem with essentially the same linear objective function, but a convex feasible region. Let $P \subseteq [0,1]^n$ denote the feasible region of the linear programming relaxation of an initial 0,1 integer programming problem. In our convex relaxation approach,

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we are hoping to construct a tractable representation of the convex hull of integer points in P, i.e., the *integer hull* of P

$$P_I := \operatorname{conv}\left(P \cap \{0,1\}^n\right).$$

However, it is impossible to efficiently find a tractable description of P_I for a general P (unless $\mathcal{P} = \mathcal{NP}$). So, in many cases we may have to be content with tractable convex relaxations that are not exact (i.e. strict supersets of the integer hull of P).

Lift-and-project methods provide an organized way of generating a sequence of convex relaxations of P which converge to the integer hull P_I of P in at most n rounds. Minimum number of rounds required to obtain the integer hull by a lift-and-project operator Γ is called the *Γ-rank* of P. The computational success of lift-and-project methods on some combinatorial optimization problems and various applications is relatively well documented (starting with the theoretical foundations in Balas' work in the 1970s [1]; this later appeared as [2]), and the majority of these computational successes come from lift-and-project methods which generate polyhedral relaxations. While many lift-and-project methods utilize in addition positive semidefiniteness constraints which in theory help generate tighter relaxations of P_I, the underlying convex optimization problems require significantly more computational resources to solve, and are prone to run into more serious numerical stability issues. Therefore, before committing to the usage of a certain lift-and-project method, it would be wise to understand the conditions under which the usage of additional computational resources would be well justified. Indeed, this argument applies to any collection of lift-and-project operators that trade off quality of approximation with computational resources (time, memory, etc.) required. That is, to utilize the strongest operators, one needs a better understanding of the class of problems on which these strongest operators' computational demands will be worthwhile in the returns they provide.

In the next section, we introduce a number of known lift-and-project operators and some of their basic properties, with the focus being on the following operators (all of which utilize positive semidefiniteness constraints):

- SA_+ (see [3,4]), a positive semidefinite variant of the Sherali–Adams operator SA defined in [5];
- Las, due to Lasserre [6];
- BZ'_+ (see [3,4]), a strengthened version of the Bienstock–Zuckerberg operator BZ_+ [7].

Then, in Section 3, we look into some elementary polytopes which represent some basic situations in 0,1 integer programs. We consider two families of polytopes: unit hypercubes that are *chipped* or *cropped* to various degrees of severity. First, given an integer $n \geq 1$ and a real number ρ where $0 \leq \rho \leq n$, the *chipped hypercube* is defined to be

$$P_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq n - \rho \right\}.$$

Similarly, we define the *cropped hypercube*

$$Q_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i \in S} (1 - x_i) + \sum_{i \notin S} x_i \geq \rho,\ \forall S \subseteq [n] \right\},$$

where $[n]$ denotes the set $\{1, \ldots, n\}$. These two families of polytopes have been shown to be bad instances for many lift-and-project methods and cutting-plane procedures (see, among others, [8–17]). Moreover, these elementary sets are interesting in many other contexts as well. For instance, note that each constraint defining $Q_{n,\rho}$ removes a specific extreme point of the unit hypercube from the feasible region. In many 0,1 integer programming problems and in 0,1 mixed integer programming problems, such *exclusion* constraints are relatively commonly used.

Herein, we show that these sets are also bad instances for the strongest known operators, extending the previously known results in this vein. In particular, we show the following: