ARTICLE IN PRESS

Discrete Optimization ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Optimization

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Anchored rectangle and square packings[☆,☆☆]

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ARTICLE INFO

Article history:
Received 5 August 2016
Received in revised form 13 August 2017
Accepted 21 August 2017
Available online xxxx

Keywords:
Rectangle packing
Anchored rectangle
Greedy algorithm
Charging scheme
Approximation algorithm

ABSTRACT

For points $p_1, \dots, p_n$ in the unit square $[0,1]^2$, an *anchored rectangle packing* consists of interior-disjoint axis-aligned empty rectangles $r_1, \dots, r_n \subseteq [0,1]^2$ such that point p_i is a corner of the rectangle r_i (that is, r_i is *anchored* at p_i) for $i = 1, \dots, n$. We show that for every set of n points in $[0,1]^2$, there is an anchored rectangle packing of area at least $7/12 - O(1/n)$, and for every $n \in \mathbb{N}$, there are point sets for which the area of every anchored rectangle packing is at most $2/3$. The maximum area of an anchored *square* packing is always at least $5/32$ and sometimes at most $7/27$.

The above constructive lower bounds immediately yield constant-factor approximations, of $7/12 - \varepsilon$ for rectangles and $5/32$ for squares, for computing anchored packings of maximum area in $O(n \log n)$ time. We prove that a simple greedy strategy achieves a $9/47$-approximation for anchored square packings, and $1/3$ for lower-left anchored square packings. Reductions to maximum weight independent set (MWIS) yield a QPTAS for anchored rectangle packings in $\exp(\mathrm{poly}(\varepsilon^{-1} \log n))$ time and a PTAS for anchored square packings in $n^{O(1/\varepsilon)}$ time.

1. Introduction

Let $P = \{p_1, \dots, p_n\}$ be a finite set of points in an axis-aligned bounding rectangle U. An *anchored rectangle packing* for P is a set of axis-aligned empty rectangles $r_1, \dots, r_n$ that lie in U, are interior-disjoint, and p_i is one of the four corners of r_i for $i = 1, \dots, n$ (a rectangle is *empty* if it does not contain any point from P in its interior); rectangle r_i is said to be *anchored* at p_i.

For a given point set $P \subset U$, we wish to find the maximum total area $A(P)$ of an anchored rectangle packing of P. Since the ratio between areas is an affine invariant, we may assume that $U = [0,1]^2$. However,

☆ An extended abstract of this paper appeared in the Proceedings of the 32nd International Symposium on Computational Geometry, June 2016, Leibniz International Proceedings in Informatics (LIPIcs) series, Schloss Dagstuhl; http://dx.doi.org/10.4230/LIPIcs.SoCG.2016.13 .

☆☆ This work was partially supported by the NSF awards CCF-1422311 and CCF-1423615.

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http://dx.doi.org/10.1016/j.disopt.2017.08.003
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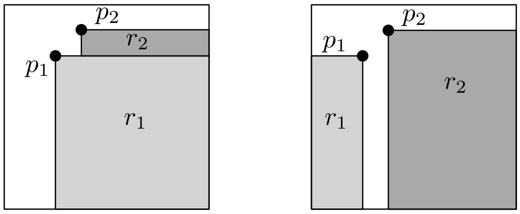

Fig. 1. For $P = \{p_1, p_2\}$, with $p_1 = (\frac{1}{4}, \frac{3}{4})$ and $p_2 = (\frac{3}{8}, \frac{7}{8})$, a greedy algorithm selects rectangles of area $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{5}{8} = \frac{41}{64}$ (left), which is less than the area $\frac{1}{4} \cdot \frac{3}{4} + \frac{5}{8} \cdot \frac{7}{8} = \frac{47}{64}$ of the packing on the right.

Table 1
Table of results for the four variants studied in this paper. The last two columns refer to lower-left anchored rectangles and lower-left anchored squares, respectively.

Anchored packing with	rectangles	squares	LL-rect.	LL-sq.
Guaranteed max. area	$\frac{7}{12} - O(\frac{1}{n}) \leq A(n) \leq \frac{2}{3}$	$\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$	0	0
Greedy approx. ratio	$7/12 - \varepsilon$	9/47	0.091 [2]	1/3
Approximation scheme	QPTAS	PTAS	QPTAS	PTAS

if we are interested in the maximum area of an *anchored square packing*, we *must* assume that $U = [0, 1]^2$ (or that the aspect ratio of U is bounded from below by a constant; otherwise, with an *arbitrary* rectangle U, the guaranteed area is only zero).

Finding the maximum area of an anchored rectangle packing of n given points is suspected but not known to be NP-hard. Balas and Tóth [1] showed that the number of distinct rectangle packings that attain the maximum area, $A(P)$, can be exponential in n. From the opposite direction, the same authors [1] proved an exponential upper bound on the number of maximum area configurations, namely $2^n C_n = \Theta(8^n/n^{3/2})$, where $C_n = \frac{1}{n+1}\binom{2n}{n} = \Theta(4^n/n^{3/2})$ is the nth Catalan number. Note that a greedy strategy may fail to find $A(P)$; see Fig. 1.

Variants and generalizations. We consider three additional variants of the problem. An *anchored square packing* is an anchored rectangle packing in which all rectangles are squares; a *lower-left anchored rectangle packing* is a rectangle packing where each point $p_i \in r_i$ is the lower-left corner of r_i; and a *lower-left anchored square packing* has both properties. We suspect that all variants, with rectangles or with squares, are NP-hard. Here, we put forward several approximation algorithms, while it is understood that the news regarding NP-hardness can occur at any time or perhaps take some time to establish.

The problem can be generalized to other geometric shapes with distinct representatives. Let $P = \{p_1, \ldots, p_n\}$ be a finite set of points in a compact domain $U \subset \mathbb{R}^d$, and let $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ be n families of measurable sets (e.g., rectangles, squares, or disks) such that for all $r \in \mathcal{F}_i$, we have $p_i \in r \subseteq U$ and $\mu(r) \geq 0$ is the measure of r. An *anchored packing for* $(P, \mathcal{F})$ is a set of pairwise interior-disjoint representatives $r_1, \ldots, r_n$ with $r_i \in \mathcal{F}_i$ for $i = 1, \ldots, n$. We wish to find an anchored packing for $(P, \mathcal{F})$ of maximum measure $\sum_{i=1}^{n} \mu(r_i)$. While some variants are trivial (e.g., when $U = [0, 1]^2$ and $\mathcal{F}_i$ consists of all rectangles containing p_i), there are many interesting and challenging variants (e.g., when $\mathcal{F}_i$ consists of disks containing p_i; or when U is nonconvex). In this paper we assume that the domain U and the families $\mathcal{F}$ are axis-aligned rectangles or axis-aligned squares in the plane.

Contributions. Our results are summarized in Table 1.

(i) We first deduce upper and lower bounds on the maximum area of an anchored rectangle packing of n points in $[0, 1]^2$. For $n \in \mathbb{N}$, let $A(n) = \inf_{|P|=n} A(P)$. We prove that $\frac{7}{12} - O(1/n) \leq A(n) \leq \frac{2}{3}$ for all $n \in \mathbb{N}$ (Sections 2 and 3).

(ii) Let $A_{\text{sq}}(P)$ be the maximum area of an anchored square packing for a point set P, and $A_{\text{sq}}(n) = \inf_{|P|=n} A_{\text{sq}}(P)$. We prove that $\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$ for all n (Sections 2 and 4).

Please cite this article in press as: K. Balas, et al., Anchored rectangle and square packings, Discrete Optimization (2017), http://dx.doi.org/10.1016/j.disopt.2017.08.003.