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Stable Marriage and Roommates problems with restricted edges: Complexity and approximability[☆]

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ABSTRACT

In the Stable Marriage and Roommates problems, a set of agents is given, each of them having a strictly ordered preference list over some or all of the other agents. A matching is a set of disjoint pairs of mutually acceptable agents. If any two agents mutually prefer each other to their partner, then they block the matching, otherwise, the matching is said to be stable. We investigate the complexity of finding a solution satisfying additional constraints on restricted pairs of agents. Restricted pairs can be either *forced* or *forbidden*. A stable solution must contain all of the forced pairs, while it must contain none of the forbidden pairs.

Dias et al. (2003) gave a polynomial-time algorithm to decide whether such a solution exists in the presence of restricted edges. If the answer is no, one might look for a solution close to optimal. Since optimality in this context means that the matching is stable and satisfies all constraints on restricted pairs, there are two ways of relaxing the constraints by permitting a solution to: (1) be blocked by as few as possible pairs, or (2) violate as few as possible constraints n restricted pairs.

Our main theorems prove that for the (bipartite) Stable Marriage problem, case (1) leads to NP-hardness and inapproximability results, whilst case (2) can be solved in polynomial time. For non-bipartite Stable Roommates instances, case (2) yields an NP-hard but (under some cardinality assumptions) 2-approximable problem. In the case of NP-hard problems, we also discuss polynomially solvable special cases, arising from restrictions on the lengths of the preference lists, or upper bounds on the numbers of restricted pairs.

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1. Introduction

In the classical *Stable Marriage problem* (SM) [1], a bipartite graph is given, where one colour class symbolises a set of men U and the other colour class stands for a set of women W. Man u and woman w are connected by edge uw if they find one another mutually acceptable. Each participant provides a strictly ordered preference list of the acceptable agents of the opposite gender. An edge uw *blocks* matching M if it is not in M, but each of u and w is either unmatched or prefers the other to their partner. A *stable matching* is a matching not blocked by any edge. From the seminal paper of Gale and Shapley [1], we know that the existence of such a stable solution is guaranteed and one can be found in linear time. Moreover, the solutions form a distributive lattice [2]. The two extreme points of this lattice are called the *man-* and *woman-optimal stable matchings* [1]. These assign each man/woman their best partner reachable in any stable matching. Another interesting and useful property of stable solutions is the so-called Rural Hospitals Theorem. Part of this theorem states that if an agent is unmatched in one stable matching, then all stable solutions leave him unmatched [3].

One of the most widely studied extensions of SM is the *Stable Roommates problem* (SR) [1,4], defined on general graphs instead of bipartite graphs. The notion of a blocking edge is as defined above (except that it can now involve any two agents in general), but several results do not carry over to this setting. For instance, the existence of a stable solution is not guaranteed any more. On the other hand, there is a linear-time algorithm to find a stable matching or report that none exists [4]. Moreover, the corresponding variant of the Rural Hospitals Theorem holds in the roommates case as well: the set of matched agents is the same for all stable solutions [5]. We summarise this observation as follows:

Theorem 1.1 (*Gusfield and Irving [5]*). *Given an instance of* SR*, the same set of agents is matched in all stable matchings.*

Both SM and SR are widely used in various applications. In markets where the goal is to maximise social welfare instead of profit, the notion of stability is especially suitable as an optimality criterion [6]. For SM, the oldest and most common area of applications is employer allocation markets [7]. On one side, job applicants are represented, while the job openings form the other side. Each application corresponds to an edge in the bipartite graph. The employers rank all applicants to a specific job offer and similarly, each applicant sets up a preference list of jobs. Given a proposed matching M of applicants to jobs, if an employer–applicant pair exists such that the position is not filled or a worse applicant is assigned to it, and the applicant received no contract or a worse contract, then this pair blocks M. In this case the employer and applicant find it mutually beneficial to enter into a contract outside of M, undermining its integrity. If no such blocking pair exists, then M is stable. Stability as an underlying concept is also used to allocate graduating medical students to hospitals in many countries [8]. SR on the other hand has applications in the area of P2P networks [9].

Forced and forbidden edges in SM and SR open the way to formulate various special requirements on the sought solution. Such edges now form part of the extended problem instance: if an edge is *forced*, it must belong to a constructed stable matching, whilst if an edge is *forbidden*, it must not. In certain market situations, a contract is for some reason particularly important, or to the contrary, not wished by the majority of the community or by the central authority in control. In such cases, forcing or forbidding the edge and then seeking a stable solution ensures that the wishes on these specific contracts are fulfilled while stability is guaranteed. Henceforth, the term *restricted edge* will be used to refer either to a forbidden edge or a forced edge. The remaining edges of the graph are referred as *unrestricted edges*.

Note that simply deleting forbidden edges or fixing forced edges and searching for a stable matching on the remaining instance does not solve the problem of finding a stable matching with restricted edges. Deleted edges (corresponding to forbidden edges, or those adjacent to forced edges) can block that matching. Therefore, to meet both requirements on restricted edges and stability, more sophisticated methods are needed.