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Joint mixability of some integer matrices

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ABSTRACT

We study the problem of permuting each column of a given matrix to achieve minimum maximal row sum or maximum minimal row sum, a problem of interest in probability theory and quantitative finance where quantiles of a random variable expressed as the sum of several random variables with unknown dependence structure are estimated. If the minimum maximal row sum is equal to the maximum minimal row sum the matrix has been termed *jointly mixable* (see e.g. Haus (2015), Wang and Wang (2015), Wang et al. (2013)). We show that the lack of joint mixability (the joint mixability gap) is not significant, i.e., the gap between the minimum maximal row sum and the maximum minimal row sum is either zero or one for a class of integer matrices including binary and complete consecutive integers matrices. For integer matrices where all entries are drawn from a given set of discrete values, we show that the gap can be as large as the difference between the maximal and minimal elements of the discrete set. The aforementioned result also leads to a polynomial-time approximation algorithm for matrices with restricted domain. Computing the gap for a $\{0,1,2\}$-matrix is proved to be equivalent to finding column permutations minimizing the difference between the maximum and minimum row sums. A polynomial procedure for computing the optimum difference by solving the maximum flow problem on an appropriate graph is given.

1. Introduction and background

We consider the problem of permuting each column of a given matrix to achieve minimum maximal row sum or maximum minimal row sum, a problem of recent interest in quantitative finance (see e.g., [1–3]) where quantiles of a random variable expressed as the sum of several random variables with unknown dependence structure are estimated. If the minimum maximal row sum is equal to the maximum minimal row sum the matrix is termed *jointly mixable*, a notion first introduced in [4] for general families of probability distributions. In this paper inspired by the recent work of Haus [2], we develop the study of joint mixability

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of some classes of integer matrices in a novel direction: we focus on the lack of mixability. After a brief introduction, we start in Section 2 with a result concerning binary matrices where we establish that the lack of joint mixability is not significant, i.e., the gap between the minimum maximal row sum and the maximum minimum row sum is either zero or one. Since a necessary and sufficient condition for joint mixability of binary matrices is known, one can immediately conclude that the gap is equal to one if the condition fails. Besides, optimal permutations achieving the gap can be obtained in linear time. A similar conclusion holds for complete consecutive integer matrices, a class of integer matrices defined in [2], as we establish in Section 3. In a generalization to integer matrices where all entries are drawn from a given set of discrete values, we show in Section 4 that the gap between the optimized minimum and maximum row sums can be as large as the difference between the minimal and maximal elements of the discrete set. This observation leads to a polynomial time approximation algorithm. For matrices with values from the set $\{0, 1, 2\}$ (termed *two-ary* matrices) we prove in Section 5 that computing the gap is equivalent to finding column permutations minimizing the difference between the maximum and minimum row sums. We also describe a polynomial-time procedure to compute the difference between the minimum maximal row sum and the maximum minimal row sum using so-called *swap operations* that can be implemented within a maximum-flow algorithm for an appropriately defined capacitated network for a given problem instance. Finally, in Section 6 we hint at the challenging nature of the problem by showing that the linear programming bound is trivial.

It is hoped that the present paper will spark renewed interest in the discrete mathematics community for this fascinating problem of considerable importance in statistics and quantitative finance. An attractive feature of the present paper is the elementary nature of proofs which rely on simple combinatorial arguments and involve at most some linear programming.

There is already considerable research activity on the continuous space counterpart of the problems of this paper, see e.g., the recent papers [5–7,4]. The standard reference on joint mixability in probability theory is [7]. In an atomless probability space, where random variables X_i which are distributed according to given univariate distributions F_i, $i = 1, \dots, d$ are given, one asks the question to determine whether a given distribution F is a possible distribution for the sum $S = X_1 + \cdots + X_d$. While the problem has a long history (see e.g., the introduction of [7]) Wang and Wang [7] partially answer the above question recently using the theory of *joint mixability*. In their context a vector $(X_1, \dots, X_d)$ is a joint mix if $X_1 + \cdots + X_d$ is almost surely a constant. A d-tuple of distributions $(F_1, \dots, F_d)$ is said to be *jointly mixable* if there exists a joint mix with univariate marginal distributions $F_1, \dots, F_d$. Ref. [7] reformulates the original question above in terms of the equivalent question of determining joint mixability of an n-tuple of distributions.

In the present paper we are concerned with the problem of determining *joint mixability* of matrices, which is described as the following pair of optimization problems: given a matrix $A \in \mathbb{R}^{m\times d}$, (a) find independently a permutation for each column of A such that the maximal row sum of the resulting matrix is minimized and (b) find independently a permutation for each column of A such that the minimal row sum of the resulting matrix is maximized. Let the permutations of the d columns of A be denoted as a *permutation system* $\Pi = (\pi_1, \dots, \pi_d)$, and refer to as A^Π the permuted matrix obtained by permuting column k by permutation matrix π_k, for $k = 1, \dots, d$. Hence the optimization problems we are interested in are:

$$\gamma(A) = \min_{\Pi} \max_{i=1,\dots,m} \left\{ \sum_{j=1}^{d} A^{\Pi}_{ij} \right\}, \tag{1}$$

and

$$\beta(A) = \max_{\Pi} \min_{i=1,\dots,m} \left\{ \sum_{j=1}^{d} A^{\Pi}_{ij} \right\}. \tag{2}$$

When $\gamma(A) = \beta(A)$, the matrix A is said to be *jointly mixable*, and the resulting permuted matrix is termed a *joint mix*. Naturally, the matrix problems treated in the present paper are discrete space analogs of the