Operations Research Letters 46 (2018) 487–491

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

An FPTAS for the knapsack problem with parametric weights

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ARTICLE INFO

Article history:
Received 1 February 2018
Received in revised form 26 May 2018
Accepted 13 July 2018
Available online 18 July 2018

Keywords:
Knapsack problems
Parametric optimization
Approximation algorithms

ABSTRACT

In this paper, we investigate the parametric weight knapsack problem, in which the item weights are affine functions of the form $w_i(\lambda) = a_i + \lambda \cdot b_i$ for $i \in \{1, \ldots, n\}$ depending on a real-valued parameter λ. The aim is to provide a solution for all values of the parameter. It is well-known that any exact algorithm for the problem may need to output an exponential number of knapsack solutions. We present the first fully polynomial-time approximation schemes (FPTASs) for the problem that, for any desired precision $\varepsilon \in (0, 1)$, compute $(1-\varepsilon)$-approximate solutions for all values of the parameter. Among others, we present a strongly polynomial FPTAS running in $\mathcal{O}\left(\frac{n^5}{\varepsilon^2} \cdot \log^3 \frac{n}{\varepsilon} \Big/ \log n\right)$ time and a weakly polynomial FPTAS with a running time of $\mathcal{O}\left(\frac{n^3}{\varepsilon^2} \cdot \log^2 \frac{n}{\varepsilon} \cdot \log P \cdot \log M \Big/ \log n\right)$, where P is an upper bound on the optimal profit and $M := \max\{|W|, n \cdot \max\{|a_i|, |b_i| : i \in \{1, \ldots, n\}\}\}$ for a knapsack with capacity W.

1. Introduction

The knapsack problem is one of the most fundamental combinatorial optimization problems: Given a set of n items with weights and profits and a knapsack capacity, the task is to choose a subset of the items with a maximum profit such that the weight of these items does not exceed the knapsack capacity. The problem is known to be weakly $\mathcal{NP}$-hard and solvable in pseudo-polynomial time. Moreover, several constant factor approximation algorithms and approximation schemes have been developed for the problem [11,13,14,16,17] (cf. [15] for an overview).

In this paper, we investigate a generalization of the problem in which the weights are no longer constant but affine functions depending on a parameter $\lambda \in \mathbb{R}$. More precisely, for a knapsack with *capacity* W and for each *item* i in the *item set* $\{1, \ldots, n\}$ with *profit* $p_i \in \mathbb{N}_{>0}$, the *weight* w_i is now of the form $w_i(\lambda) := a_i + \lambda \cdot b_i$ with $a_i, b_i \in \mathbb{Z}$. The resulting optimization problem can be stated as follows:

$$p^*(\lambda) = \max \sum_{i=1}^{n} p_i \cdot x_i$$
$$\sum_{i=1}^{n} (a_i + \lambda \cdot b_i) \cdot x_i \leqslant W$$
$$x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\}.$$

The aim of this *parametric weight knapsack problem* is to return a partition of the real line into intervals $(-\infty, \lambda_1], [\lambda_1, \lambda_2], \ldots, [\lambda_{k-1}, \lambda_k], [\lambda_k, +\infty)$ together with a solution x^* for each interval such that this solution is optimal for all values of λ in the interval.

Besides the fact that the parametric weight knapsack problem is clearly $\mathcal{NP}$-hard to solve since it contains the traditional knapsack problem, Woeginger has shown that any exact algorithm for the problem may need to return an exponential number of knapsack solutions in general (Ref. [16] in [2]). In this paper, we are interested in a *fully polynomial time approximation scheme* for the parametric weight knapsack problem. We will show that, for any desired precision $\varepsilon \in (0, 1)$, a polynomial number of intervals suffices in order to be able to provide a $(1-\varepsilon)$-approximate solution for each $\lambda \in \mathbb{R}$.

In the following, we let $P \leqslant \sum_{i=1}^{n} p_i$ denote an upper bound on the optimal profit and set

$$M := \max\{|W|, n\} \cdot \max\{|a_i|, |b_i| : i \in \{1, \ldots, n\}\}. \quad (1)$$

1.1. Previous work

A large number of publications investigated parametric versions of well-known problems. This includes the parametric shortest path problem [4,12,21,23], the parametric minimum spanning tree problem [1,7], the parametric maximum flow problem [8,18,22], and the parametric minimum cost flow problem [3] (cf. [9] for an overview).

A problem that is related to the parametric weight knapsack problem considered here is the parametric *profit* knapsack problem, in which the weights are constant but the profits take on an

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https://doi.org/10.1016/j.orl.2018.07.005

affine form $p_i(\lambda) = a_i + \lambda \cdot b_i$. Carstensen [3] first showed that the number of breakpoints of the optimal profit function may be exponentially large in general and pseudo-polynomially large in the case of integral input data. Eben-Chaime [5] later presented a pseudo-polynomial exact algorithm for this problem. Giudici et al. [9] derived a PTAS for the general problem and, for the case that $\lambda \geqslant 0$ and $a_i, b_i \geqslant 0$ for all $i \in \{1, \ldots, n\}$, an FPTAS with a weakly polynomial running-time. Recently, Holzhauser and Krumke [10] presented an FPTAS for the problem that works for arbitrary integral values of a_i, b_i and runs in strongly polynomial time.

To the best of our knowledge, the parametric weight knapsack problem considered here was mentioned in only one publication so far. Burkard and Pferschy [2] show that the optimal profit of the parametric weight knapsack problem can attain an exponential number of values in general, yielding that any exact algorithm for the problem must output an exponential number of knapsack solutions. The authors present a pseudo-polynomial algorithm for the *inverse* problem, in which the largest value for the parameter is searched that allows some given profit.

1.2. Our contribution

We present the first FPTASs for the parametric weight knapsack problem. In fact, these are the first approximation algorithms for the problem and it is even the first algorithmic approach at all. Our algorithms are based on an implicit scaling technique due to Erlebach et al. [6], in which a $(1 + \varepsilon)$-grid is laid over the search space of the underlying dynamic programming scheme such that only a polynomial number of entries must be evaluated. By refining this approach, we are able to derive several FPTASs for the parametric weight knapsack problem. Our best strongly polynomial FPTAS achieves a running time of

$$\mathcal{O}\left(\frac{n^5}{\varepsilon^2} \cdot \log^3 \frac{n}{\varepsilon} \Big/ \log n\right).$$

Moreover, one of our weakly polynomial FPTASs runs in

$$\mathcal{O}\left(\frac{n^3}{\varepsilon^2} \cdot \log^2 \frac{n}{\varepsilon} \cdot \log P \cdot \log M \Big/ \log n\right)$$

time, where P is an upper bound on the optimal profit and $M := \max\{|W|, n \cdot \max\{|a_i|, |b_i| : i \in \{1, \ldots, n\}\}\}$ for a knapsack with capacity W.

1.3. Organization

The results of this paper are divided into three main parts. In Section 2, we develop a very first version of our FPTAS. First, in Section 2.1, we recapitulate the FPTAS for the traditional knapsack problem due to Erlebach et al. [6] and generalize it to the parametric setting in Sections 2.2 and 2.3. In Section 3, we show how we can improve this algorithm in order to obtain a strongly polynomial FPTAS for the problem. Finally, we present another possible improvement in Section 4, which yields a faster weakly polynomial FPTAS for the problem.

2. Obtaining a parametric FPTAS

2.1. Traditional FPTAS

Consider the case of some fixed value for λ such that the weights have a constant (and possibly negative) value w_i. The FPTAS due to Erlebach et al. [6] for the traditional knapsack problem is based on a well-known dynamic programming scheme, which was originally designed to solve the problem exactly in pseudo-polynomial time: Let P denote an upper bound on the maximum profit of a solution to the given instance. For $k \in \{0, \ldots, n\}$ and $p \in \{0, \ldots, P\}$, let $w(k, p)$ denote the minimum weight that is necessary in order to obtain a profit of exactly p with the first k items. For $k = 0$, we set $w(0, p) = 0$ for $p = 0$ and $w(0, p) = W + 1$ for $p > 0$. For $k \in \{1, \ldots, n\}$ and for the case that $p_k \leqslant p$, we compute the values $w(k, p)$ recursively by

$$w(k, p) = \min\{w(k-1, p), w(k-1, p-p_k) + w_k\}, \tag{2}$$

representing the choice to either not pack the item or to pack it, respectively. Else, if $p_k > p$, we set $w(k, p) = w(k-1, p)$. The largest value of p such that $w(n, p) \leqslant W$ then reveals the optimal solution to the problem. The procedure runs in pseudo-polynomial time $\mathcal{O}(nP)$.

The idea of the FPTAS is to perform this dynamic programming scheme only for a polynomial number of profits: Instead of considering each integral profit in $\{0, \ldots, P\}$, the profit space is being reduced to the points in the set

$$S := \{0\} \cup \left\{(1+\varepsilon)^{\frac{i}{n}} : i \in \{0, \ldots, \lceil n \log_{1+\varepsilon} P \rceil\}\right\}.$$

The entries of $w(\cdot, \cdot)$ are only computed for the points in the set S such that the evaluation of the dynamic programming scheme now takes polynomial time $\mathcal{O}\left(n^2 \cdot \log_{1+\varepsilon} P\right) = \mathcal{O}\left(\frac{n^2}{\varepsilon} \cdot \log P\right)$. In order for the recursion to be well-defined, the term $p - p_k$ is "rounded up" to the next value in S in Eq. (2). Hence, $w(k, p)$ denotes the minimum weight that is necessary to achieve a profit of *at least* p. It is easy to see that, in the case of constant weights, this approach yields an FPTAS for the knapsack problem (cf. [6]).

2.2. Parametric FPTAS

Now consider the parametric problem setting. Obviously, the weights of the items change as the parameter λ increases such that new solutions may become feasible and current solutions may become infeasible as the parameter λ increases. The idea of the FPTAS is to consider each possible profit $p \in S$ individually and to determine the values of λ for which a profit of p can be achieved by a feasible solution to the scaled knapsack instance.

In the following, it will be useful to interpret the underlying dynamic programming scheme as a shortest path problem: Consider an acyclic graph similar to the one shown in Fig. 1 and let $(p^{(0)}, \ldots, p^{(\bar{P})})$ denote an ordered sequence of the elements in S. For each $k \in \{0, \ldots, n\}$ and profit $p^{(l)} \in S$, we insert a node $v_{l,k}$ and, if $k \leqslant n-1$, connect it with $v_{l,k+1}$ via an edge with zero length as well as with $v_{l',k+1}$ via an edge with length $w_{k+1}(\lambda)$, where l' is the largest value such that $p^{(l')} \leqslant p^{(l)} + p_{k+1}$. It is easy to see that the structure of the graph reflects the recursion given in (2) applied to the set S. An approximate solution to the knapsack instance then corresponds to the largest profit $p \in S$ that allows a path from $v_{0,0}$ to $v_{p,n}$ with length at most W.

Now consider some specific profit $p \in S$. In general, there may be a super-polynomial number of paths in the graph that lead from $v_{0,0}$ to $v_{p,n}$. The lengths of these paths are described by affine functions depending on the parameter λ and may increase or decrease as λ increases. Since we are interested in shortest paths, we can restrict our considerations to the function $\omega^{(p)} : \mathbb{R} \to \mathbb{R}$ mapping the parameter λ to the length of a shortest path from $v_{0,0}$ to $v_{p,n}$. Clearly, the function $\omega^{(p)}$ is concave, continuous, and piecewise linear since it is the point-wise minimum of finitely many affine functions (see Fig. 2). Whenever $\omega^{(p)}(\lambda) \leqslant W$, the knapsack solution induced by a shortest path from $v_{0,0}$ to $v_{p,n}$ is feasible and attains a profit of at least p. Besides the special cases that, for all $\lambda \in \mathbb{R}$, $\omega^{(p)}(\lambda) \leqslant W$ or, for all $\lambda \in \mathbb{R}$, $\omega^{(p)} > W$, there is at most one interval $I_-^{(p)} := (-\infty, \lambda_1]$ and at most one interval $I_+^{(p)} := [\lambda_2, +\infty)$ with $\omega^{(p)}(\lambda_1) = \omega^{(p)}(\lambda_2) = W$ containing the values of