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Error bounds for augmented truncation approximations of continuous-time Markov chains

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ABSTRACT

This paper considers the augmented truncation approximation of the generator of an ergodic continuous-time Markov chain with a countably infinite state space. The main purpose of this paper is to present bounds for the absolute difference between the stationary distributions of the original generator and its augmented truncation. As examples, we apply the bounds to an $M/M/s$ retrial queue and an upper Hessenberg Markov chain.

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1. Introduction

Let $\{X(t); t \ge 0\}$ denote an ergodic continuous-time Markov chain on state space $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, which has the generator $\boldsymbol{Q} := (Q(i,j))_{i,j\in\mathbb{Z}_+}$. Let $\boldsymbol{\pi} := (\pi(i))_{i\in\mathbb{Z}_+}$ denote the stationary distribution vector of $\boldsymbol{Q}$, i.e., $\boldsymbol{\pi Q} = \boldsymbol{0}$ and $\boldsymbol{\pi e} = 1$, where $\boldsymbol{e} = (1, 1, \dots)^\top$.

In this paper, we consider the augmented truncation approximation of $\boldsymbol{Q}$. To this end, we introduce the *northwest-corner truncation* of $\boldsymbol{Q}$. For $n \in \mathbb{Z}_+$, let ${}_{(n)}\boldsymbol{Q} := ({}_{(n)}Q(i,j))_{i,j\in\mathbb{Z}_n}$ denote the $(n+1)\times(n+1)$ northwest-corner truncation of $\boldsymbol{Q}$, i.e., ${}_{(n)}Q(i,j) = Q(i,j)$ for $i,j \in \mathbb{Z}_n$, where $\mathbb{Z}_n = \{0, 1, \dots, n\}$.

Note that ${}_{(n)}\boldsymbol{Q}$ is a Q-matrix (diagonally dominant matrix with nonnegative off-diagonal elements and nonpositive row sums; see [1, Section 2.1, page 64]). The ergodicity of $\boldsymbol{Q}$ implies that ${}_{(n)}\boldsymbol{Qe} \le \boldsymbol{0}, \ne \boldsymbol{0}$, i.e., ${}_{(n)}\boldsymbol{Q}$ is not *conservative* (see [1, Section 2.1, page 64]). Hence, we *augment* the elements of ${}_{(n)}\boldsymbol{Q}$ and extend the truncated state space $\mathbb{Z}_n$ to the original one $\mathbb{Z}_+$ in order to construct a conservative Q-matrix as an *approximation* to the original generator $\boldsymbol{Q}$.

For $n \in \mathbb{Z}_+$, we define ${}_{(n)}\tilde{\boldsymbol{Q}} := ({}_{(n)}\tilde{Q}(i,j))_{i,j\in\mathbb{Z}_+}$ as a conservative Q-matrix such that, for $i \in \mathbb{Z}_+$,

$$ {}_{(n)}\tilde{Q}(i,j) = Q(i,j) + \psi_{n,i}(j) \sum_{\ell \ge n+1, \ell \ne i} Q(i,\ell), \quad j \in \mathbb{Z}_n, \tag{1a} $$

$$ {}_{(n)}\tilde{Q}(i,j) = Q(i,i), \qquad i = j \in \overline{\mathbb{Z}}_n, \tag{1b} $$

$$ {}_{(n)}\tilde{Q}(i,j) = 0, \qquad i \ne j \in \overline{\mathbb{Z}}_n, \tag{1c} $$

where $\overline{\mathbb{Z}}_n = \{m \in \mathbb{Z}_+ : m \ge n+1\}$, and where $\psi_{n,i}(\cdot)$ is a probability distribution on $\mathbb{Z}_n$ that may depend on $(n,i) \in \mathbb{Z}_+^2$. Clearly, for any fixed $(i,j) \in \mathbb{Z}_+^2$, we have $\lim_{n\to\infty} {}_{(n)}\tilde{Q}(i,j) = Q(i,j)$. Thus, we refer to ${}_{(n)}\tilde{\boldsymbol{Q}}$ as the *n-th order augmented northwest-corner truncation approximation* (called the *augmented truncation*, for short) of $\boldsymbol{Q}$.

In defining ${}_{(n)}\tilde{\boldsymbol{Q}}$, we append additional elements to ${}_{(n)}\boldsymbol{Q}$ in a such way that the resulting Q-matrix ${}_{(n)}\tilde{\boldsymbol{Q}}$ has the same order (size) as the original generator $\boldsymbol{Q}$. This is aimed at performing algebraic operations involving the original generator $\boldsymbol{Q}$, e.g., ${}_{(n)}\tilde{\boldsymbol{Q}} - \boldsymbol{Q}$.

Such an extension (of the truncated state space $\mathbb{Z}_n$) is not unique. However, (1a)–(1c) imply that the appended states in $\overline{\mathbb{Z}}_n$ are transient and that all the closed communicating classes of ${}_{(n)}\tilde{\boldsymbol{Q}}$ are finite sets in $\mathbb{Z}_n$. Therefore, ${}_{(n)}\tilde{\boldsymbol{Q}}$ has at least one stationary distribution vector, whose elements in $\overline{\mathbb{Z}}_n$ are all equal to zero.

We denote by ${}_{(n)}\tilde{\boldsymbol{\pi}} := ({}_{(n)}\tilde{\pi}(i))_{i\in\mathbb{Z}_+}$, an arbitrary one of the stationary distribution vectors of ${}_{(n)}\tilde{\boldsymbol{Q}}$. For later use, let $|\cdot|$ denote a matrix (resp. vector) obtained by taking the absolute value of each element of the matrix (resp. vector) in the vertical bars. For any row vector $\boldsymbol{\mu} := (\mu(i))_{i\in\mathbb{Z}_+}$ and nonnegative column vector

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$\boldsymbol{v} := (v(i))_{i\in\mathbb{Z}_+} \neq \mathbf{0}$, let $\|\boldsymbol{\mu}\|_{\boldsymbol{v}} = \sum_{i\in\mathbb{Z}_+}|\mu(i)|v(i) = |\boldsymbol{\mu}|\boldsymbol{v}$, which denotes the $\boldsymbol{v}$-norm of the row vector $\boldsymbol{\mu}$.

The main purpose of this paper is to derive bounds for $\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}}$, where $\boldsymbol{g} := (g(i))_{i\in\mathbb{Z}_+}$ denotes a nonnegative column vector. For this purpose, we assume the following.

Assumption 1.1 (*$\boldsymbol{f}$-Modulated Drift Condition*). There exist some $b \in (0,\infty)$, finite set $\mathbb{C} \subset \mathbb{Z}_+$, column vectors $\boldsymbol{v} := (v(i))_{i\in\mathbb{Z}_+} > \mathbf{0}$ and $\boldsymbol{f} := (f(i))_{i\in\mathbb{Z}_+} \geq \boldsymbol{e}$ such that $v_0 := \inf_{i\in\mathbb{Z}_+} v(i) > 0$ and

$$\boldsymbol{Q}\boldsymbol{v} \leq -\boldsymbol{f} + b\mathbf{1}_{\mathbb{C}}, \tag{2}$$

where, for any set $\mathbb{B} \subseteq \mathbb{Z}_+$, $\mathbf{1}_{\mathbb{B}} := (1_{\mathbb{B}}(i))_{i\in\mathbb{Z}_+}$ denotes a column vector such that $1_{\mathbb{B}}(i) = 1$ for $i \in \mathbb{B}$ and $1_{\mathbb{B}}(i) = 0$ for $i \in \mathbb{Z}_+ \setminus \mathbb{B}$.

There have been many studies on the augmentation approximation (see e.g., [3,4,17,20]). Masuyama [13] studied the *last-column-block-augmented truncation* (called *LC-block-augmented truncation*, for short) of continuous-time block monotone Markov chains (see [6]). Note that the LC-block-augmented truncation is a special case of the augmented truncation $_{(n)}\tilde{\boldsymbol{Q}}$. Masuyama [13] presented some tractable bounds for the total variation distance $\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{e}}$ (see [11,12] for discrete-time block-monotone Markov chains and also see [8,19] for their special cases).

In addition, Masuyama [14] studied the LC-block-augmented truncation without either any transition structure (such as block monotonicity) or any specific ergodicity of $\boldsymbol{Q}$. However, the fundamental bounds for $\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}}$ presented in [14] include an intractable factor $\boldsymbol{\pi}\boldsymbol{v}$ (see Theorems 2.1 and 2.2 therein).

Recently, Liu and Li [9] derived a bound for $\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}}$ without the factor $\boldsymbol{\pi}\boldsymbol{v}$ under an additional condition that Assumption 1.1 holds for $\mathbb{C} = \{i_0\} \in \mathbb{Z}_+$ and $\boldsymbol{v}$ having nondecreasing elements (see Theorem 6.1 therein).

The main contribution of this paper is to present such bounds without $\boldsymbol{\pi}\boldsymbol{v}$ in more general settings. The rest of this paper is divided into two sections. Section 2 presents the main results of this paper. To illustrate their applicability, Section 3 considers an $M/M/s$ retrial queue and an upper Hessenberg Markov chain.

2. Main results

2.1. Basic bounds

Let $\mathrm{sgn}(\cdot)$ denote the sign function, i.e., $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(x) = x/|x|$ for $x \neq 0$. Let $\tilde{\boldsymbol{g}} := (\tilde{g}(i))_{i\in\mathbb{Z}_+}$ denote a column vector such that

$$\tilde{g}(i) = \mathrm{sgn}(_{(n)}\tilde{\pi}(i) - \pi(i)) \cdot g(i), \quad i \in \mathbb{Z}_+.$$

We then prove the following lemma.

Lemma 2.1. *Under Assumption 1.1, we have*

$$\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}} = {}_{(n)}\tilde{\boldsymbol{\pi}}(_{(n)}\tilde{\boldsymbol{Q}} - \boldsymbol{Q})\boldsymbol{h}, \qquad n \in \mathbb{Z}_+, \tag{3}$$

where $\boldsymbol{h} := (h(i))_{i\in\mathbb{Z}_+}$ denotes a solution of the following Poisson equation (see, e.g., [5]):

$$-\boldsymbol{Q}\boldsymbol{h} = \tilde{\boldsymbol{g}} - (\boldsymbol{\pi}\tilde{\boldsymbol{g}})\boldsymbol{e}. \tag{4}$$

Proof. Using (4), $_{(n)}\tilde{\boldsymbol{\pi}}\boldsymbol{e} = 1$ and $_{(n)}\tilde{\boldsymbol{\pi}}_{(n)}\tilde{\boldsymbol{Q}} = \mathbf{0}$, we have

$$\begin{aligned} {}_{(n)}\tilde{\boldsymbol{\pi}}(_{(n)}\tilde{\boldsymbol{Q}} - \boldsymbol{Q})\boldsymbol{h} &= -{}_{(n)}\tilde{\boldsymbol{\pi}}\boldsymbol{Q}\boldsymbol{h} = {}_{(n)}\tilde{\boldsymbol{\pi}}\{\tilde{\boldsymbol{g}} - (\boldsymbol{\pi}\tilde{\boldsymbol{g}})\boldsymbol{e}\} \\ &= (_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi})\tilde{\boldsymbol{g}} = |_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}|\,\boldsymbol{g}, \end{aligned}$$

which yields (3).

For $j \in \mathbb{Z}_+$, we define $\boldsymbol{h}_j := (h_j(i))_{i\in\mathbb{Z}_+}$ as a column vector such that, for $i \in \mathbb{Z}_+$,

$$h_j(i) = \mathsf{E}_i\left[\int_0^{\tau_j} \tilde{g}(X(t))\mathrm{d}t\right] - (\boldsymbol{\pi}\tilde{\boldsymbol{g}})\mathsf{E}_i[\tau_j], \tag{5}$$

where $\tau_j = \inf\{t \geq 0 : X(t) = j\}$ for $j \in \mathbb{Z}_+$ and $\mathsf{E}_i[\,\cdot\,] = \mathsf{E}[\,\cdot \mid X(0) = i]$ for $i \in \mathbb{Z}_+$. Note that $\boldsymbol{h}_j$ is a solution of the Poisson equation (4) (see [14, Lemma B.2]). Thus, Lemma 2.1 implies that

$$\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}} = {}_{(n)}\tilde{\boldsymbol{\pi}}(_{(n)}\tilde{\boldsymbol{Q}} - \boldsymbol{Q})\boldsymbol{h}_j, \quad (n,j) \in \mathbb{Z}_+^2. \tag{6}$$

We now introduce some definitions to bound $|\boldsymbol{h}_j|$. For $\beta > 0$, let $\boldsymbol{\Phi}^{(\beta)} = (\phi^{(\beta)}(i,j))_{i,j\in\mathbb{Z}_+}$ denote

$$\boldsymbol{\Phi}^{(\beta)} = \int_0^\infty \beta e^{-\beta t}\boldsymbol{P}^{(t)}\mathrm{d}t, \tag{7}$$

where $\boldsymbol{P}^{(t)} := (P^{(t)}(i,j))_{i,j\in\mathbb{Z}_+}$ is the transition matrix function of the Markov chain $\{X(t)\}$ with generator $\boldsymbol{Q}$, i.e., $P^{(t)}(i,j) = \mathsf{P}(X(t) = j \mid X(0) = i)$ for $i,j \in \mathbb{Z}_+$. Note here that $\boldsymbol{\Phi}^{(\beta)} > \mathbf{O}$ is a stochastic matrix due to the ergodicity of $\boldsymbol{Q}$. For any finite set $\mathbb{C} \subset \mathbb{Z}_+$, let $m_{\mathbb{C}}^{(\beta)}$ denote a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{Z}_+)$ of $\mathbb{Z}_+$ such that, for $j \in \mathbb{Z}_+$,

$$m_{\mathbb{C}}^{(\beta)}(j) := m_{\mathbb{C}}^{(\beta)}(\{j\}) = \min_{i\in\mathbb{C}} \phi^{(\beta)}(i,j) > 0. \tag{8}$$

The following lemma presents a bound for $|\boldsymbol{h}_j|$.

Lemma 2.2. *If Assumption 1.1 holds, then, for $j \in \mathbb{Z}_+$,*

$$|\boldsymbol{h}_j| \leq \kappa^{(\boldsymbol{g})}\left(\boldsymbol{v} + \frac{b}{\beta m_{\mathbb{C}}^{(\beta)}(j)}\boldsymbol{e}\right), \quad \mathbf{0} \leq \boldsymbol{g} \leq \boldsymbol{f},\ \beta > 0, \tag{9}$$

where

$$\kappa^{(\boldsymbol{g})} = 1 + \frac{\boldsymbol{\pi}\boldsymbol{g}}{\inf_{\ell\in\mathbb{Z}_+} f(\ell)}. \tag{10}$$

Proof. It follows from (5), $|\tilde{\boldsymbol{g}}| \leq \boldsymbol{g} \leq \boldsymbol{f}$ and $\boldsymbol{f} \geq \boldsymbol{e}$ that

$$\begin{aligned} |h_j(i)| &\leq \mathsf{E}_i\left[\int_0^{\tau_j} f(X(t))\mathrm{d}t\right] + \boldsymbol{\pi}\boldsymbol{g}\mathsf{E}_i[\tau_j] \\ &\leq \left(1 + \frac{\boldsymbol{\pi}\boldsymbol{g}}{\inf_{\ell\in\mathbb{Z}_+} f(\ell)}\right)\mathsf{E}_i\left[\int_0^{\tau_j} f(X(t))\mathrm{d}t\right] \\ &= \kappa^{(\boldsymbol{g})}\mathsf{E}_i\left[\int_0^{\tau_j} f(X(t))\mathrm{d}t\right], \qquad i,j \in \mathbb{Z}_+, \end{aligned} \tag{11}$$

where the last equality holds due to (10). Following the derivation of the bound (2.17) in [14], we can prove that

$$\mathsf{E}_i\left[\int_0^{\tau_j} f(X(t))\mathrm{d}t\right] \leq v(i) + \frac{b}{\beta m_{\mathbb{C}}^{(\beta)}(j)}, \quad i,j \in \mathbb{Z}_+.$$

Substituting this into (11) yields (9). □

Remark 2.1. From (2) and $\mathbf{0} \leq \boldsymbol{g} \leq \boldsymbol{f}$, we have $\boldsymbol{\pi}\boldsymbol{g} \leq \boldsymbol{\pi}\boldsymbol{f} \leq b\sum_{i\in\mathbb{C}}\pi(i) \leq b$. Thus, if $\boldsymbol{\pi}\boldsymbol{g}$ in (10) is intractable, then it can be replaced by b or $b\sum_{i\in\mathbb{C}}\pi(i)$.

We now arrive at the main theorem of this paper.

Theorem 2.1. *Suppose that Assumption 1.1 holds. Let*

$$\overline{\phi}_{\mathbb{C}}^{(\beta)} = \sup_{j\in\mathbb{Z}_+}\min_{i\in\mathbb{C}} \phi^{(\beta)}(i,j) = \sup_{j\in\mathbb{Z}_+} m_{\mathbb{C}}^{(\beta)}(j). \tag{12}$$

We then have, for $n \in \mathbb{Z}_+$,

$$\|_{(n)}\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_{\boldsymbol{g}} \leq \kappa^{(\boldsymbol{g})} \cdot E_0^{(\beta)}(n), \quad \mathbf{0} \leq \boldsymbol{g} \leq \boldsymbol{f},\ \beta > 0, \tag{13}$$

where $\kappa^{(\boldsymbol{g})}$ is given in (10) and

$$E_0^{(\beta)}(n) = {}_{(n)}\tilde{\boldsymbol{\pi}}\,|_{(n)}\tilde{\boldsymbol{Q}} - \boldsymbol{Q}|\left(\boldsymbol{v} + \frac{b}{\beta\overline{\phi}_{\mathbb{C}}^{(\beta)}}\boldsymbol{e}\right). \tag{14}$$

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