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Directed Graphical Structure, Nash Equilibrium, and Potential Games

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Abstract

This paper considers the directed graphical structure of a game, called *influence structure*, where a directed edge from player i to player j indicates that player i may be able to affect j's payoff via his unilateral change of strategies. We give a necessary and sufficient condition for the existence of pure-strategy Nash equilibrium of games having a directed graph in terms of the structure of that graph. We also discuss the relationship between the structure of graphs and potential games.

Keywords: Directed graphical games, Pure-strategy Nash equilibrium, Potential games
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1. Introduction

A recurrent theme in game theory is the study of properties of games – and, in particular, of their equilibria – that can be extracted from partial information about the players' strategies and payoff functions (e.g., [39], [31]) . Since a basic assumption in game theory is that each player has certain preferences among the outcomes while her payoff may be influenced by the choices of her own as well as the opponents' ([28], p.1), a simple and natural example of such information is that who is influenced by whom. This paper introduces the directed graphical structure of a game, called *influence structure*, where a directed edge from player i to player j indicates that player i may be able to affect j's payoff via his unilateral change of strategies. We study the relationship of the structure of the directed graph with the existence of pure-strategy Nash equilibrium of games having that graph and with potential games. Theorem 1 shows that, given a directed graph, all games having it have pure-strategy Nash equilibrium if and only if the graph does not contain a *reflexive cycle*. Here by reflexive cycle we mean a cycle containing more than one node where each node has a self-loop. Theorem 2 states that a game having a graph without reflexive cycle is a generalized ordinal potential game, but may not be an ordinal potential one.

Graphical structures and their relationship with properties of games have long been studied (see [16], [17]). The seminal paper [23] introduced graphical games to describe direct influences between players in games and investigated its relationship with equilibria and their algorithms. Based on it, various studies have been done on how to search for (pure- or mixed-strategy) Nash equilibria on graphs and their computational complexity ([32], [25], [12], [22], [7]). Though graphical games are assumed to have an underlying undirected graph (a few exceptions, e.g., [42], treated the graph as directed), it does not necessarily mean that the influences are symmetric. The interpretation can be that for two players connected by an edge, only one of them influences the payoffs of the other. Nevertheless, by adopting directed graph, the asymmetric influence structure can be treated more explicitly. Also, directed graphical structure helps to study nodes (players) of the graph (game) that has (or lack) self-loops, that is, players who can (or cannot) ever influence his own payoff by switching to a different strategy. Those help to study the structure and computational properties of equilibria of games. Asymmetry and self-loops play important roles in the results of this paper. Further, it will provide insights on important topics such as asymmetric follow on Facebook and Twitter (see [18], [24], [37]), learned helplessness ([41]) and atomization ([38]) in social psychology, which deeply affect people's thinking and behavior in the information age.

Also, the idea of relating the influence of players' choice with the properties of a game is not entirely new. In the literature of social choice theory, a concept called effectivity function (EFF) was developed to describe players' power on the outcomes of a game form ([1], [30], [29], [35], [2], [3], [36]). In terms of EFFs, [11] characterized acceptability (the existence of pure-strategy Nash equilibrium for any preference profile for a game form and every corresponding outcome is Pareto effecient) and dominance-solvability of game forms, and [14] characterized the existence of pure-strategy Nash equilibrium in 2-person game forms. The difference is that EFF is defined by the power a group of players on the outcome rather than on a player's payoff, which makes it more suitable to capture groups' blocking and dominating in social choice situations than to do with an individual's decision making. Also, an EFF depends on the decision rule of the game form which assigns to each strategy profile an outcome, while an I-structure is defined within a general framework directly based on players' payoffs. Those differences lead to different subjects and focuses of EFF and influence structure.

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