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On the complexity of instationary gas flows

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ABSTRACT

We study a simplistic model of instationary gas flows consisting of a sequence of k stationary gas flows. We present efficiently solvable cases and NP-hardness results, establishing complexity gaps between stationary and instationary gas flows (already for $k = 2$) as well as between instationary gas s-t-flows and instationary gas b-flows.

1. Introduction

This paper studies the algorithmic complexity of time-varying flows in gas transport networks. In the gas transport literature, these flows are called *instationary* in contrast to *stationary* gas flows that describe a steady state situation. This paper presents efficiently solvable problems and identifies complexity gaps between stationary and instationary gas flows, as well as between instationary gas flows with a single source/sink and multi-terminal instationary gas flows. Our ultimate goal is to contribute to a better understanding of the particular difficulty of instationary gas flows. To this end, we introduce a simple model of instationary gas flows in Section 3, and present an efficiently solvable instationary gas flow problem in Section 4, examples of more complicated scenarios in Section 5, and finally an NP-hardness result in Section 6.

2. Stationary gas flows

Before turning to the more general case of instationary gas flows, we introduce some basic facts about stationary gas flows. In contrast to classical network flows where, within given capacity bounds, flow may be distributed throughout a network ad libitum, gas flows are governed by the laws of physics. Essentially, in a gas network the (stationary) flow along an arc (pipeline) is uniquely determined by the pressures at the two endpoints of the arc. For an in-depth treatment of (stationary) flows in gas networks we refer to the recent book [4]. The simplest and most widely adapted model for stationary gas flows is Weymouth's equation [7]: For an arc $a = (u, v)$, the flow value x_a along a satisfies

$$\beta_a x_a\,|x_a| = \pi_u - \pi_v, \tag{1}$$

where the node potentials $\pi_u = {p_u}^2$ and $\pi_v = {p_v}^2$ are the squared pressures at nodes u and v, respectively, and $\beta_a > 0$ is a given constant specifying the resistance of arc a. Here, a negative flow value x_a on arc $a = (u, v)$ represents flow in the opposite direction from node v to node u. This stationary gas flow model forms the basis of this paper.

Consider a directed graph G with node set V and arc set A. For given node balances $b \in \mathbb{R}^V$ with $\sum_{v\in V} b_v = 0$, a stationary gas flow satisfying supplies and demands given by b can be computed by solving the following convex min-cost b-flow problem [2,5],

$$\begin{aligned} \min \quad & \sum_{a\in A} \frac{\beta_a}{3}|x_a|^3 \\ \text{s.t.} \quad & \sum_{a\in\delta^{\text{out}}(v)} x_a - \sum_{a\in\delta^{\text{in}}(v)} x_a = b_v \quad \forall v \in V, \end{aligned} \tag{2}$$

where the objective function is chosen such that the resulting KKT conditions yield Weymouth's equations (1). The corresponding dual program is

$$\max_{\pi}\Bigg(\sum_{u\in V} b_u\pi_u - 2\sum_{(u,v)\in A}\frac{|\pi_u-\pi_v|^{3/2}}{3\sqrt{\beta_{(u,v)}}}\Bigg). \tag{3}$$

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Strong duality holds and the dual variables yield the node potentials in (1) which are unique up to translation by an arbitrary value. Problems (2) and (3) can be solved efficiently within arbitrary precision.

Throughout this paper, we assume that there are uniform bounds on all node potentials given by an interval $[\pi_{\min}, \pi_{\max}]$. A stationary gas flow x with corresponding node potential $\pi \in \mathbb{R}^V$ is *feasible* if $\pi_{\min} \leq \pi_v \leq \pi_{\max}$ for all $v \in V$. We state an important theorem on stationary gas flows, which essentially follows from the work of Calvert and Keady [1] (see also [3]), and for which we give a short proof for the sake of completeness.

Theorem 1 (*[1]*). *Consider a network with source s, sink t, and potential interval* $[\pi_{\min}, \pi_{\max}]$. *If* $\beta_a \geq \beta'_a > 0$ *for all* $a \in A$, *then the value of a maximal feasible stationary gas s-t-flow for arc resistances* β *is at most the value of a maximal feasible stationary gas s-t-flow for arc resistances* β'.

Proof. For some s-t-flow value $B = b_s = -b_t \geq 0$ (and $b_v = 0$ for $v \in V \setminus \{s, t\}$), consider the primal problem (2) and the dual problem (3). By combining (3), (1), and (2), the value z^* of an optimal solution (x^*, π^*) satisfies

$$z^* = \sum_{v\in V} b_v\, \pi_v^* - 2\sum_{a\in A} \frac{\beta_a}{3} |x_a^*|^3 = B(\pi_s^* - \pi_t^*) - 2z^*,$$

and thus $\pi_s^* - \pi_t^* = 3z^*/B$. Notice that $\pi_s^* \geq \pi_v^* \geq \pi_t^*$ for all $v \in V$ due to (1). In particular, there is a feasible stationary gas s-t-flow of value B if and only if $3z^*/B \leq \pi_{\max} - \pi_{\min}$ (note that the dual solution π^* is unique up to translation). As the optimal value z^* of (2) is an increasing function of the arc resistances β_a, $a \in A$, the existence of a feasible stationary gas s-t-flow of value B for arc resistances β thus implies the existence of such a flow for arc resistances $\beta' \leq \beta$. □

3. A simple instationary gas flow model

We introduce a model of instationary gas flows that, while being simple enough to allow for a theoretical analysis, still captures essential characteristics and exhibits interesting properties. In particular, we prove meaningful results that constitute an interesting first step in explaining the increased difficulty of instationary versus stationary gas flows.

For $k \in \mathbb{Z}_{>0}$, a *k-stage gas flow* x is a k-tuple $(x^1, \ldots, x^k)$ of stationary gas flows (where we interpret $x^1, \ldots, x^k$ as a temporal succession). If x^i satisfies supplies and demands $b^i \in \mathbb{R}^V$, $i = 1, \ldots, k$, then x in total satisfies supplies and demands $b = b^1 + \cdots + b^k \in \mathbb{R}^V$ and is called *k-stage gas b-flow*. For two distinguished nodes $s, t \in V$, x is a *k-stage gas s-t-flow* of *value q* if it satisfies supplies and demands $b \in \mathbb{R}^V$ with $b_s = -b_t = q$ and $b_v = 0$ for $v \in V \setminus \{s, t\}$. A k-stage gas flow x is called *stationary* if $x^1 = \cdots = x^k$, otherwise x is called *instationary*. Finally, a k-stage gas flow x is *feasible* if $x^1, \ldots, x^k$ are feasible stationary gas flows.

Remark 2. Notice that, in marked contrast to actual gas transport, in our model there is no correlation between consecutive flows x^i and x^{i+1}. Moreover, the model allows to arbitrarily buffer or borrow flow in each node (i.e., flow may be withdrawn or injected at each node) at each stage as long as the accumulated node balances $b^1 + \cdots + b^k$ add up to the desired b (cp. examples in Section 5).

We study the following two algorithmic problems for $k \in \mathbb{Z}_{>0}$: first, the *maximum k-stage gas s-t-flow problem*, whose input is a network G with source $s \in V$, sink $t \in V$, and interval $[\pi_{\min}, \pi_{\max}]$, and the task is to find a feasible k-stage gas s-t-flow of maximum value; second, the *k-stage gas b-flow problem*, whose input is a network G with supplies and demands $b \in \mathbb{R}^V$, as well as interval $[\pi_{\min}, \pi_{\max}]$, and the task here is to find a feasible k-stage gas b-flow.

4. Maximum 2-stage gas s-t-flows

We first show that there exists an efficiently computable stationary maximum 2-stage gas s-t-flow.

Theorem 3. *Taking two copies of the maximum feasible stationary gas s-t-flow yields an optimal solution to the maximum 2-stage gas s-t-flow problem.*

Consider a feasible 2-stage gas s-t-flow (x^1, x^2) with node potentials $\pi^1, \pi^2 \in \mathbb{R}^V$. By definition, the flow $\tilde{x} := \frac{1}{2}(x^1 + x^2)$ is an s-t-flow (not necessarily a stationary gas flow induced by node potentials π, though), and the value of the feasible 2-stage gas s-t-flow (x^1, x^2) is exactly twice the value of $\tilde{x}$.

Lemma 4. *The node potentials* $\bar{\pi} := \frac{1}{2}(\pi^1 + \pi^2)$ *induce a feasible stationary gas flow* $\bar{x}$ *with* $\operatorname{sgn}(\bar{x}_a) = \operatorname{sgn}(\tilde{x}_a)$ *and* $|\bar{x}_a| \geq |\tilde{x}_a|$ *for each* $a \in A$.

Proof. By definition of $\bar{x}$ and x^i, $i = 1, 2$, we have

$$\bar{x}_a = \operatorname{sgn}(\bar{\pi}_u - \bar{\pi}_v)\sqrt{|\bar{\pi}_u - \bar{\pi}_v|}/\sqrt{\beta_a},$$
$$x_a^i = \operatorname{sgn}(\pi_u^i - \pi_v^i)\sqrt{|\pi_u^i - \pi_v^i|}/\sqrt{\beta_a},$$

for each arc $a = (u, v) \in A$. Moreover, by definition of $\bar{\pi}$, we get $\bar{\pi}_u - \bar{\pi}_v = \big((\pi_u^1 - \pi_v^1) + (\pi_u^2 - \pi_v^2)\big)/2$. The lemma thus follows from the next observation. □

Observation 5. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(\sigma) = \operatorname{sgn}(\sigma)\sqrt{|\sigma|}$. *Then, for all* $\sigma^1, \sigma^2 \in \mathbb{R}$,

$$\operatorname{sgn}\Big(f\Big(\frac{\sigma^1+\sigma^2}{2}\Big)\Big) = \operatorname{sgn}\Big(\frac{f(\sigma^1)+f(\sigma^2)}{2}\Big),$$
$$\Big|f\Big(\frac{\sigma^1+\sigma^2}{2}\Big)\Big| \geq \Big|\frac{f(\sigma^1)+f(\sigma^2)}{2}\Big|.$$

Proof. Notice that $f(-\sigma) = -f(\sigma)$ for all $\sigma \in \mathbb{R}$ (in particular, $f(0) = 0$), and $f|_{\mathbb{R}_{\geq 0}}$ is non-negative, strictly increasing, and concave. Therefore the statement is clear for the cases that σ^1 and σ^2 are both non-negative or both non-positive.

It remains to consider the case $\sigma^1 < 0 < \sigma^2$. The equality statement on the signs is an immediate consequence of f's properties noted above. By symmetry we may assume that $|\sigma^1| \leq \sigma^2$ such that $\frac{1}{2}(\sigma^1 + \sigma^2) \geq 0$ and thus $f\big(\frac{1}{2}(\sigma^1 + \sigma^2)\big) \geq 0$. By concavity of $f|_{\mathbb{R}_{\geq 0}}$, we get two inequalities:

$$f\Big(\frac{\sigma^1+\sigma^2}{2}\Big) \geq \frac{f(0)+f(\sigma^1+\sigma^2)}{2} = \frac{f(\sigma^1+\sigma^2)}{2},$$

$$f(\sigma^1+\sigma^2) - f(\sigma^1) = f(-|\sigma^1| + \sigma^2) + f(|\sigma^1|) \geq f(\sigma^2) + f(0) = f(\sigma^2).$$

The latter inequality implies that $f(\sigma^1 + \sigma^2) \geq f(\sigma^1) + f(\sigma^2)$. Together with the former inequality this yields the desired result. □

It follows from Lemma 4 and (1) that by increasing the β_a values individually for each arc $a \in A$, we arrive at a network where the s-t-flow $\tilde{x}$ is a feasible stationary gas s-t-flow induced by the node potentials $\bar{\pi} := \frac{1}{2}(\pi^1 + \pi^2)$. More precisely, we need to set $\tilde{\beta}_a := \beta_a \bar{x}_a^2/\tilde{x}_a^2 \geq \beta_a$. Thus, by Theorem 1, the value of the stationary maximal feasible gas s-t-flow x^* in the network with original arc resistances β is at least the value of $\tilde{x}$, which is half the value of our feasible 2-stage gas s-t-flow (x^1, x^2). Summarizing, the value of the feasible 2-stage gas s-t-flow (x^*, x^*) is at least the value of (x^1, x^2). This concludes the proof of Theorem 3.