Operations Research Letters 46 (2018) 453–456

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On the contracts between doctors and rural hospitals

Beatriz Millán [a], Eliana Pepa Risma [b,*,1]

[a] Universidad Nacional de San Juan - Instituto de Matemática Aplicada San Luis, Argentina
[b] Universidad Nacional de San Luis - Instituto de Matemática Aplicada San Luis, Argentina

ARTICLE INFO

Article history:
Received 20 March 2018
Received in revised form 1 June 2018
Accepted 1 June 2018
Available online 15 June 2018

Keywords:
Matching
Contracts
Stability
Rural hospital

ABSTRACT

We prove that the set of doctors assigned to a hospital with unfilled positions is the same in all stable allocations for a many-to-one matching model with contracts where all hospitals have q-separable preferences. However, the characteristics of the relationships among these agents may differ from one stable allocation to another.

1. Introduction

The many-to-one matching with contracts model, introduced in [4], embeds several models which are relevant to the design of real-world institutions. These models include, among others, the classical matching models (without contracts) introduced in [1] and [5].

The original version of the rural hospital theorem states that the set of single agents is the same in all stable matchings. This was shown in [7] for the special case of the classical one-to-one matching model, in which all agents find every agent on the opposite side of the market acceptable. Extensions of this result were provided in [9] and [2]. Later on, an additional finding was proved in [10] for a many-to-one matching model without contracts where all hospitals have q-responsive preferences: any hospital that fails to fill out all of its positions at some stable matching, will obtain the same assignment, i.e., the same doctors, under every stable matching. [6] generalized this result for a many-to-one matching model without contracts where all hospitals have substitutable and q-separable preferences. This property ensures that the hospitals with spare capacity cannot blame the matching-maker or clearinghouse for this situation.

In this paper, we show that the latter version of the rural hospital theorem partially extends to a many-to-one matching model with contracts where the hospitals have substitutable and q-separable preferences. More specifically, we prove that every hospital that has unfilled positions at some stable allocation signs contracts with the same set of doctors at every stable allocation, although the contracts may differ from one stable allocation to another. Then, when contracts are incorporated into the matching model, the agents still cannot blame the matching-maker for the numerical and qualitative maldistribution of doctors among hospitals since any change in the procedure yielding a stable allocation has no effect on this issue. Observe that those agents could complain about the contractual conditions obtained in different stable allocations, but it does not seem to be a serious problem in practice (it could eventually be addressed later as an internal question). Nevertheless, the last finding has a significant theoretical implication. This is, the extension of results from classical matching models to matching models with contracts needs to be mathematically validated in order to avoid inaccuracies or false claims, as we argue in the concluding remark of this work.

2. Preliminaries

We begin by describing the many-to-one matching model with contracts introduced in [4]. The market is bilateral, there is a finite set of agents partitioned into a set of doctors D and a set of hospitals H. There is also a finite set of contracts $\mathbf{X}$. Each contract $x \in \mathbf{X}$ names one doctor $x_D \in D$ and one hospital $x_H \in H$. The allocation problem consists of selecting which contracts are concluded. In this way, it is specified which agents are assigned to each other and, at the same time, their contractual conditions are established. Unlike in models without contracts, the characteristics of the relationship among two agents (salary, schedules, work tasks, etc.) are not fixed beforehand. In fact, $\mathbf{X}$ may contain different contracts involving the same pair of agents $(d, h) \in \mathbf{D} \times \mathbf{H}$. Every matching market

* Corresponding author.
E-mail address: ebpepa@unsl.edu.ar (E. Pepa Risma).
1 Instituto de Matemática Aplicada San Luis, Universidad Nacional de San Luis, Italia 1556 - San Luis (5700) República, Argentina.

https://doi.org/10.1016/j.orl.2018.06.001
0167-6377/

without contracts can be addressed as a special matching market with contracts where $\mathbf{X}$ contains one and only one contract for each pair $(d, h) \in \mathbf{D} \times \mathbf{H}$.

Given a set of contracts $Y \subseteq \mathbf{X}$ and an agent $j \in D \cup H$, we denote by Y_j the set of all contracts involving j which are contained in Y.

For every set of contracts $Y \subseteq \mathbf{X}$ we denote:

$$D(Y) := \{d \in D : Y_d \neq \varnothing\}$$

In the many-to-one model that we consider here, every doctor is allowed to sign the maximum of one contract. An allocation is a subset of contracts meeting this requirement. Formally:

Definition 1. A set of contracts $Z \subseteq \mathbf{X}$ is an **allocation** if $x \neq y$ implies $x_D \neq y_D$ for all $x, y \in Z$.

Observe that the empty set is an allocation.

Given a set of contracts $Y \subseteq \mathbf{X}$, let $A(Y)$ denote the set of all allocations which are subsets of Y.

Each agent $j \in D \cup H$ has an antisymmetric, transitive and complete preference relation $\succ_j$ over $A(\mathbf{X}_j)$, the set of all allocations which consist of contracts naming j exclusively. Observe that for all $d \in D$ and all $Z \in A(\mathbf{X}_d)$ we have $|Z| \leq 1$. A profile of preferences $\mathbf{P}$ is a set consisting of one preference relation per agent.

A particular matching market with contracts is a pair $(\mathbf{X}, \mathbf{P})$ where $\mathbf{X}$ is the set of all existent contracts and $\mathbf{P}$ is the profile of preferences.

Frequently, for a given set, it is necessary to consider its first ranked subset according to the preferences of a particular agent. Observe that the mentioned subset must be an allocation due to how the preferences are defined. It could be the empty allocation.

Definition 2. For any set of contracts $Y \subseteq \mathbf{X}$ and any agent $j \in D \cup H$, the **choice set of** j **given** Y is

$$C_j(Y) = \max_{\succ_j} A(Y_j).$$

Definition 3. Given a set of contracts $Y \subseteq \mathbf{X}$ and an agent $j \in D \cup H$, the set of **rejected-by-j contracts belonging to** Y is

$$R_j(Y) = Y_j - C_j(Y).$$

Notation 4. *Given a set of contracts $Y \subseteq \mathbf{X}$, we denote $C_D(Y) = \bigcup_{d \in D} C_d(Y)$ and $C_H(Y) = \bigcup_{h \in H} C_h(Y)$. We call $C_D(Y)$ and $C_H(Y)$ the* ***choice set of all doctors given*** *Y and the* ***choice set of all hospitals given*** *Y respectively.*

An allocation could contain unwanted contracts. Since the agents are allowed to cancel them unilaterally, the absence of these contracts is a necessary condition to obtain stable allocations.

Definition 5. The allocation $Y \in A(\mathbf{X})$ is **individually rational** if $C_D(Y) = C_H(Y) = Y$.

Whenever two agents wish to sign a contract, they are free to do it and they are also free to terminating previous contracts. Consequently, an allocation can be blocked by a contract.

Definition 6. Given $Y \in A(\mathbf{X})$, the contract $x \in \mathbf{X} - Y$ is a **blocking contract** for Y if

$$x \in C_{x_D}(Y \cup \{x\}) \cap C_{x_H}(Y \cup \{x\}).$$

Now, we summarize the concept of stability as follows.

Definition 7. $Y \in A(\mathbf{X})$ is a **stable allocation** if
(i) Y is individually rational;
(ii) There are no blocking contracts for Y.

Let $S(\mathbf{X}, \mathbf{P})$ denote the set of all stable allocations in the particular market $(\mathbf{X}, \mathbf{P})$. In [4] it was proved that $S(\mathbf{X}, \mathbf{P}) \neq \varnothing$ if the preferences of all the hospitals satisfy substitutability, a property establishing that no contract stops being chosen because another contract stops being available.

Definition 8. The preferences of an agent $j \in D \cup H$ satisfy **substitutability** if $R_j(X) \subseteq R_j(Y)$ whenever $X_j \subseteq Y_j \subseteq \mathbf{X}$.

Under substitutability, Definition 7 is equivalent to the one introduced in [4].

Additionally, in [4] it was introduced the law of aggregate demand, a condition over the preferences establishing that the number of contracts chosen by an agent either rises or stays the same if the set of available contracts increases.

Definition 9. The preferences of an agent $j \in D \cup H$ satisfy the **law of aggregate demand** (LAD) if

$$|C_j(Y)| \leq |C_j(Z)|$$

for all $Y \subseteq Z \subseteq \mathbf{X}$.

LAD is less restrictive than the condition of q-separability introduced in [6]. In fact, if the preferences of an agent j are q-separable, they satisfy LAD trivially; however, the reciprocal does not necessarily hold true. See [8] for an illustration of the practical importance of preferences satisfying LAD which are not q-separable.

q-separability states that by adding an acceptable option without exceeding the agent's quota (maximum number of positions that it is willing to fill), a better set of contracts is obtained, whereas the opposite occurs by incorporating an unacceptable option. In terms of contracts, q-separability can be defined as follows:

Definition 10. The preferences of a hospital h are **q_h-separable** if:

(i) For all $Y \subsetneqq \mathbf{X}$ such that $|Y_h| < q_h$ and every contract x such that $x_H = h$, if there is no contract naming x_D in Y_h (h has preferences over allocations, and no allocation can contain more than one contract naming the same doctor) then:

$$Y_h \cup \{x\} \succeq_h Y_h \iff \{x\} \succeq_h \varnothing$$

(ii) $\varnothing \succeq_h Y_h$ for all allocation $Y \subseteq \mathbf{X}$ such that $|Y_h| > q_h$.

Another condition over the preferences which is used with relative frequency in classical matching theory is q-responsiveness. This condition states that, besides being q-separable, the agent's preferences over two sets differing from one other by only one element preserve the order over these non-common elements. Observe that q-responsiveness implies substitutability and, consequently, the existence of stable outcomes is guaranteed in a market where all the hospitals have q-responsive preferences. Next, we introduce the natural extension of the definition of q-responsiveness for the many-to-one matching with contracts model considered here.

Definition 11. The preferences of a hospital h are **q_h-responsive** if:

(i) they are **q_h-separable;**

(ii) for all $Y \subsetneqq \mathbf{X}$ such that $|Y_h| < q_h$ and every pair of contracts x and y such that $x_H = y_H = h$, if there is no contract naming x_D or y_D in Y_h, then:

$$Y_h \cup \{x\} \succeq_h Y_h \cup \{y\} \iff \{x\} \succeq_h \{y\}$$