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The Shapley value for the probability game

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ABSTRACT

The main goal of this paper is to introduce the probability game. On one hand, we analyze the Shapley value by providing an axiomatic characterization. We propose the so-called independent fairness property, meaning that for any two players, the player with larger individual value gets a larger portion of the total benefit. On the other, we use the Shapley value for studying the profitability of merging two agents.

1. Introduction

Game theory has been proved to be a useful tool when analyzing the ballistic missile defense budget allocation and the cooperative *R&D* profit allocation problems. A large quantity of facts has proved that missile interception [6] and cooperative *R&D* issue have always been a significant topic in the field of military tactical ballistic missile and *R&D* problems. Take the ballistic missile defense situation as an example. Once used in the war, it is sure to be anti tactical ballistic missile interceptor weapons [4]. One of the typical cases originates from the Gulf War, in which the interception of Scud missiles by Patriot missile captured worldwide attention [1]. Plenty of research [6,4,1,8] and a number of military exercises illustrate that multiple layered defense system is safer in comparison with the single one. For instance, on 2016, July, 15th, India successfully completed multi-layer ballistic missile defense system test [3]. A positive aspect that is still not yet addressed in the literature is how to distribute the defense project budget among the individual defense layers in cooperative defense situation. The most simple and direct method is to allocate the defense budget according to each layer's defense probability of success. However, it is not taken into account the fact that the defense layers are organized in one system and function as a whole. In the missile example, all the agents operate with the same aim, yielding that for any group of agents it is important that at least one succeeds, as the target is the same for all of them. Therefore, the successful action probability for the group of the agents is an important index which reflects the group action competency. By observing this, we argue that layers certainly enter into alliance and gain through cooperation because once the interceptor system fails, it can be at best costly—at worst, disastrous.

This paper broadens the game theoretic approach to the probability game, as a model of the cooperation ballistic missile defense situation, cooperative *R&D* problems and so on. We introduce the so-called probability game, of which the characteristic function is the successful acting probability of the coalition, i.e., the value of the coalition is described by the probability that at least one event in S is successful. Many concepts of allocations are proposed in the literature. Among them, we study the well known Shapley value, introduced and characterized in [7]. A mass of literature concentrating on this topic can be found to illustrate the fairness of this value, by revealing and emphasizing its properties. Examples include Shapley's efficiency, null player property, linearity and symmetry [7], Young's strong monotonicity [9], and Chun's coalitional strategic equivalence [2]. In the context of probability game, assume that players are mutually independent, we propose the so-called independent fairness property, meaning that, the player with larger successful acting probability will be assigned with more portion of the total profit. We show that for probability game, the independent fairness property can be used to characterize the Shapley value together with linearity, the dummy property and efficiency. Finally, we determine the significant threshold illustrating whether or not merger of any two players produces extra benefits to them.

The paper is organized as follows: in Section 2 we conduct on the probability game and its Shapley value. In Section 3 we deal with the characterization of the Shapley value, by proposing the independent fairness property. In Section 4 we use the Shapley value for studying the profitability of merging two agents.

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2. The Shapley value for probability game

In this section, we determine the Shapley value for the probability game, as a model of the cooperation ballistic missile defense situation, cooperative *R&D* problems, and so on.

Definition 1. The probability game is a triple (N, v, P), where N is the set of mutually independent players, $P = (p_1, p_2, \dots, p_n)$ is an n-dimension vector, with p_i the successful action probability of player i, and its characteristic function $v : 2^N \to R$ satisfying $v(\emptyset) = 0$ and for any $S \subseteq N$,

$$v(S) = \sum_{k\in S} p_k - \sum_{k,l\in S} p_k \cdot p_l + \sum_{k,l,m\in S} p_k \cdot p_l \cdot p_m - \cdots + (-1)^{s-1} \prod_{k\in S} p_k. \quad (1)$$

To interpret Definition 1, assume that each player can have a success or a failure and that successes are probabilistically independent across players. Let p_i be the probability of success of player i, then $v(S)$ is the probability that all players i in S have a success, and all other players have a failure. For the case $p_i = 0$, which means that this player is doomed to fail, then there is no need to take further consideration of this player. We exclude such case by only considering players with positive action probability.

Lots of game theoretic allocations are proposed. The most simple one is the Proportional value [5]. However, this approach ignores the fact that the players function as a congruent whole. Therefore, the players' indices for dividing the total budget should not only take into account the individual successful action probability, but also the coalitional successful action probabilities. This leads to consider the other solutions, such as the Shapley value. If we can work out the index φ_i for dividing the total budget, then we can invest $\frac{\varphi_i}{v(N)} \times W$ to each player, where W is the amount of the total budget. Next, we conduct on the determination of the Shapley value. The Shapley value, as an allocation scheme, is introduced by Shapley in 1953 as follows [7],

$$Sh_i(N, v) = \sum_{S\not\ni i, S\subseteq N} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)],\ i \in N. \quad (2)$$

Generally, because of the combinatorial terms of the Shapley value, the computation is rather hulking to deal with. Our purpose is to simplify the solution part of the probability game. Usually, the allocation based on the Shapley value supplies a distribution of the $v(N)$ among all the players according to marginal contributions with the form $v(S \cup \{i\}) - v(S), S \subseteq N \setminus \{i\}$. In the context of probability game, the next lemma denotes the characterization of the marginal contributions, playing a significant role to determine the Shapley value.

Lemma 2.1. *For any probability game (N, v, P), it holds that*

$$v(S\cup i) - v(S) = p_i(1 - v(S)) = p_i \prod_{k\in S}(1 - p_k). \quad (3)$$

The proof of (3) is trivial by the independence of the events and the probability formula.

Theorem 2.2. *(i) For the probability game, the Shapley value of player i, which implies the index for dividing the total budget, is proportional to his individual successful action probability, while inversely proportional to the other players' successful action probability. Namely,*

$$Sh_i(N, v, P) = \frac{p_i}{n} + p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \alpha_S \prod_{k\in S}(1 - p_k),\ i \in N, \quad (4)$$

where α_S is the formation probability of coalition S, which is equal to $\frac{s!(n-s-1)!}{n!}$.

(ii) Alternatively, the Shapley allocation $Sh_i(N, v, P)$ can be rewritten as

$$\begin{aligned} &Sh_i(N, v, P)\\ &= p_i - p_i \cdot [\sum_{k\in N\setminus i} \frac{1}{2} p_k - \sum_{k,l\in N\setminus i} \frac{1}{3} p_k p_l - \cdots + (-1)^n \frac{1}{n} \prod_{k\in N\setminus\{i\}} p_k] \quad \text{or}\\ &= p_i - p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{1}{|S|+1}(-1)^{|S|+1} \prod_{k\in S} p_k,\ i \in N. \end{aligned} \quad (5)$$

Proof. The validity of the theorem is due to the Shapley value applied to the marginal contribution result (3).

(i) Fix coalition $S \subseteq N, S \not\ni i$, by (3), it holds

$$\begin{aligned} Sh_i(N, v, P) &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)]\\ &= \sum_{S=\emptyset} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)]\\ &\quad + \sum_{S\not\ni i, S\neq\emptyset} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)]\\ &= \frac{p_i}{n} + p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{s!(n-s-1)!}{n!} \prod_{k\in S}(1 - p_k). \end{aligned}$$

(ii) Because of (3)

$$\begin{aligned} Sh_i(N, v, P) &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[v(S\cup i) - v(S)]\\ &= \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!} p_i - \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!} p_i \cdot v(S)\\ &= p_i - p_i \sum_{S\not\ni i} \frac{s!(n-s-1)!}{n!}[\sum_{k\in S} p_k - \sum_{k,l\in S} p_k \cdot p_l - \cdots\\ &\quad + (-1)^{s+1} \prod_{k\in S} p_k]\\ &= p_i - p_i[\sum_{k\in N\setminus\{i\}} \sum_{S\not\ni i, S\ni k} \alpha_S p_k + \cdots + (-1)^n \alpha_{N\setminus\{i\}} \prod_{k\in N\setminus\{i\}} p_k]\\ &= p_i - p_i[\sum_{k\in N\setminus\{i\}} \frac{1}{2} p_k - \sum_{k,l\in N\setminus\{i\}} \frac{1}{3} p_k p_l - \cdots + (-1)^n \frac{1}{n} \prod_{k\in N\setminus\{i\}} p_k]\\ &= p_i - p_i \sum_{|S|=1,\dots,n-1, S\not\ni i} \frac{1}{|S|+1}(-1)^{|S|+1} \prod_{k\in S} p_k. \end{aligned}$$

The last but one equality holds because for any fixed $M = \{i_1, i_2, \dots, i_m\} \subseteq N \setminus \{i\}$,

$$\begin{aligned} &\sum_{S\not\ni i, S\ni i_1, i_2, \dots, i_m} \alpha_S \cdot p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m}\\ &= \sum_{|S|=m,\dots,n-1} \frac{s!(n-s-1)!}{n!} \binom{n-m-1}{s-m} p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m}\\ &= \frac{1}{m+1} \frac{1}{C_n^{m+1}} p_{i_1} \cdot p_{i_2} \cdot \dots \cdot p_{i_m} \sum_{|S|=m,\dots,n-1} C_s^m\\ &= \frac{1}{m+1} \prod_{k\in M} p_k. \end{aligned}$$

This completes the proof of (ii). □

Although the formula is quite complicated, its economic interpretation is interesting as follows: the Shapley value index of player i is only composed of the portion of $v(N)$ relevant to