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Vector-Valued Multivariate Conditional Value-at-Risk

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ABSTRACT: In this study, we propose a new definition of multivariate conditional value-at-risk (MCVaR) as a set of vectors for arbitrary probability spaces. We explore the properties of the vector-valued MCVaR (VMCVaR) and show the advantages of VMCVaR over the existing definitions particularly for discrete random variables.

Keywords: conditional value-at-risk; multivariate risk; value-at-risk

1. Introduction *Conditional value-at-risk* (CVaR) is a widely used tool, especially in financial optimization, for assessing the risk associated with a certain decision. The value of CVaR at a confidence level p is the expected outcome given that it is unfavorable compared to at least $100p\%$ of the possible realizations. The threshold value corresponds to the p-quantile, which is also known as the *value-at-risk* (VaR). VaR is a suitable risk measure for the cases where the aim is to avoid unfavorable outcomes with high probability. However, it does not measure the magnitude of the unfavorable outcomes. To address this, CVaR has been introduced to quantify the risk as the expected value of the undesired outcomes [18].

CVaR is first introduced as a risk measure for univariate random variables. Due to its desirable properties such as coherence and law invariance, CVaR is widely incorporated into optimization problems to minimize or limit the risk of the corresponding decisions (see, e.g., [18], [19], [7]). However, in many real life problems, decision makers are interested in measuring the risk arising from multiple factors rather than a single outcome. One way of extending the univariate definitions of VaR and CVaR to the multivariate case is to use a scalarization vector for turning the random outcome vector into a scalar. However, the relative importance of criteria is usually ambiguous. The weights of criteria with respect to each other are not always quantifiable as a unique vector, as they may be subject to conflicting opinions of a group of decision makers. To address this, recent work considers a robust approach, where the set of possible scalarization vectors is assumed to be known and often assumed to be polyhedral or convex, and a worst-case scalarization vector in this set is used to scalarize the multivariate random vector [9, 11, 12, 13, 14]. These papers not only define the concept of polyhedral multivariate CVaR for finite discrete distributions, but also give optimization models and solution methods when the multivariate random outcome vector is a function of the decisions. For computability of these risk measures, the solution methods rely on sampling from the true distribution. In this context, it is natural to consider finite discrete distributions, described with a finite number of scenarios. In a related line of work, Prékopa and Lee [17] provide an alternative scalar-valued definition, where risk is measured along directions through a given reference point in the multidimensional space, and then the minimum or a combination is calculated with respect to the directions in a cone. In this paper, instead of a (scalarized) real-valued representation of the multivariate CVaR, we introduce a vector-valued risk measure without the need for the specification of an ambiguity set describing the possible weights or a reference point.

The first challenge in defining a multivariate CVaR is the determination of the p-quantile (multivariate VaR). While the p-quantile corresponds to a single real value in the univariate setting, p-quantile of a random vector may point to several vectors in the multivariate context. There are several studies addressing this issue. Prékopa [16] proposes a multivariate definition of VaR as a set of vectors, called p-Level Efficient Points (pLEPs), rather than a single value for arbitrary multivariate distributions. Cousin and Di Bernardino [3] propose an alternative approach for the continuous distribution case that computes a single vector as the expectation of the boundary surface of the multivariate quantile function. Torres et al. [21], on the other hand, introduce a direction parameter and compute the multivariate VaR vector in that direction. Di Bernardino et al. [6] and Adrian and Brunnermeier [1] also provide single, vector-valued definitions of multivariate VaR by conditioning on the information on certain criteria. As in the univariate case, these multivariate VaR definitions are only concerned with the probability of having