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A note on the problem of r disjoint (s, t)-cuts and some related issues

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Abstract

In this paper we consider the problem of finding a minimum-cost set of r disjoint (s, t)-cuts. We establish the link between this problem and some of its variations. We give a full description of the dominant of the convex hull of the incidence vectors of sets of r disjoint (s, t)-cuts. This generalizes the result for the $r = 1$ case obtained by Schrijver.

Keywords: graph partitioning, cut, polyhedra.

1. Introduction

Let $G = (V, A)$ be a directed graph with two distinguished vertices s and t connected by at least one directed path from s to t. Let r be an integer ≥ 1. An (s, t)-cut is a set of arcs of the form $\delta^+(X)$ such that $s \in X$ and $t \in V \setminus X$ (standard notation is used in the paper and recalled at the end of the section). Given a non-negative cost function c on A, the r disjoint (s, t)-cuts problem consists in computing a minimum-cost set of r disjoint (s, t)-cuts. In [9] (see [10]), Robacker proved that the maximum number of disjoint (s, t)-cuts is equal to the minimum length of an $s - t$ path. Thus, r must be less than or equal to the length of shortest path between s and t. Observe that for $r = 1$, we get the standard minimum-cut problem.

The r disjoint (s, t)-cut problem has been studied by [8] and [15, 16, 17] where it was shown to be solvable in polynomial-time. In fact, [8] studied a slightly different problem where the goal is to partition the set of vertices into $r + 1$ subsets V_0, ..., V_r such that $s \in V_0$, $t \in V_r$ and there is no arcs from V_i to V_j if $j > i+1$. The goal is then to find a partition minimizing the cost of arcs in $F = \{(uv) \in A | u \in V_i, v \in V_{i+1}, i \in \{0, 1, \ldots, r-1\}\}$. The set F can obviously be seen as a set of r disjoint (s, t)-cuts. When the arc costs are non-negative, using standard uncrossing techniques one can easily show that a minimum-cost set of r disjoint (s, t)-cuts can be found by computing a partition V_0, ..., V_r minimizing the cost of arcs in F [15].
The r disjoint (s, t)-cuts problem can also be seen as a r disjoint (s, t)-dicuts problem in an extended graph where the goal is compute a minimum cost set of r pairwise-disjoint (s, t)-dicuts where a (s, t)-dicut is a set of arcs of the form $\delta^+(X)$ such that $s \in X$, $t \in V \setminus X$ and $\delta^-(X) = \emptyset$ [16]. This can be done by building a graph G' where each arc $a = (uv)$ of G is replaced by a new node w and two arcs $\bar{a} = (uw)$ and $\hat{a} = (vw)$ such that $c_{\bar{a}} = c_a$ and $c_{\hat{a}} = 0$. Some linear formulations and polyhedral results related to the r disjoint (s, t)-dicuts problem are provided in [16, 15]. In [12], Skutella and Weber [12] study the dominant of the $s-t$-cut polytope which corresponds to the case $r = 1$. A complete description of this polyhedron is given in [10].

Let us now go back to the partitioning problem of [8] described above. This problem is generalized in [4] by integrating more specific constraints. Let $G = (V, A)$ be a directed graph, r,k, k' be positive integers, and consider $r + 1$ disjoint sets S_i, $i = 0, 1, \ldots, r$, of specified vertices (S_i can be empty). A set of arcs $F \subseteq A$ is called an $(r, k, k')-$ extended cut if there is some partition $\{V_0, V_1, \ldots, V_r\}$ of V such that:
(i) $S_i \subset V_i$, $i = 0, 1, \ldots, r$;
(ii) there is no arc from a vertex in V_i to a vertex in V_j when $j > i + k$ or $i > j + k'$;
(iii) $F = \{(uv) \in A | u \in V_i, v \in V_j,\ i, j \in \{0, 1, \ldots, r\}, i < j\}$.

Given a cost function c on A that associates with an arc $a \in A$ the cost $c_a \in \mathbb{R}$, the $(r, k, k')-$*extended cut problem* is to find an $(r, k, k')-$extended cut F such that $c(F) = \sum_{a \in F} c_a$ is minimum. This problem is a generalization of many classical combinatorial optimization problems [4].

In this paper we focus on the case $k = 1$, $k' = r$ and $c \geq 0$ which is equivalent to r disjoint (s, t)-cuts problem. Since this problem can be solved in polynomial time when the costs are non-negative, there is a chance to provide a full description of the dominant of the convex hull of the incidence vectors of sets of r disjoint (s, t)-cuts. In [10], as mentioned above, Schrijver gives such description for $r = 1$. The main contribution of our paper is to extend this result for every r.

The rest of the paper is organized as follows. In Section 2, we present linear formulations for r disjoint (s, t)-cuts problem and its variations. We give in Section 3.2 a full description of the dominant of the polytope associated with the $(r, k = 1, k' = r)$-extended cut problem which is equal to the dominant of the convex hull of the incidence vectors of sets of r disjoint (s, t)-cuts.

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