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Spitzer's identity for discrete random walks *

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Abstract

Spitzer's identity describes the position of a reflected random walk over time in terms of a bivariate transform. Among its many applications in probability theory are congestion levels in queues and random walkers in physics. We present a derivation of Spitzer's identity for random walks with bounded jumps to the left, only using basic properties of analytic functions and contour integration. The main novelty is a reversed approach that recognizes a factored polynomial expression as the outcome of Cauchy's formula.

Keywords: Spitzer's identity; fluctuation theory; transform methods; complex analysis

1 Introduction

Random walks are ubiquitous in the modern stochastics literature. This paper deals with the one-dimensional random walk on $\mathbb{Z}$ describing the partial sums $S_n := X_1 + \cdots + X_n$ ($S_0 := 0$) of i.i.d. random variables $X_1, X_2, \ldots$. The stochastic process $(S_n, n \geq 0)$, the random walk with steps X_n, arises in many areas of science to describe the evolution of certain objects subject to random fluctuations, including random walkers in physics, congestion levels in queueing theory and capital positions in insurance mathematics [9, 13]. If we indeed think of a random walk $(S_n, n \geq 0)$ as modeling capital or congestion, the monetary/physical interpretation means that large values of S_n are of particular interest, and it is natural to consider the sequence $\bar{M}_n := \max\{S_0, S_1, \ldots, S_n\}$. The study of $(\bar{M}_n, n \geq 0)$ and related quantities is referred to as the fluctuation theory of the random walk, a central topic in most classic probability textbooks [3, 7, 8, 12, 21].

Fluctuation theory became highly topical by the rise of queueing theory in the first half of the twentieth century, with foundational works of A.K. Erlang and F. Pollaczek (see the historical account in [14]) and as primary example the waiting time process in the single-server queue. Let consecutive customers arriving to a single-server queue be numbered $n = 1, 2, \ldots$. Denote by B_n the service time of customer n, and by C_n the time between the arrivals of customers n and $n+1$. Then with $X_{n+1} := B_n - C_n$, and W_n the waiting time of customer n, the waiting time process (also known as the Lindley process) is given by

$$W_{n+1} = (W_n + X_{n+1})^+, \quad n = 0, 1, \ldots, \tag{1.1}$$

*Dedicated to Prof. Onno J. Boxma on the occasion of his 65th birthday

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