Operations Research Letters 46 (2018) 199–204

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Existence and uniqueness of optimal dynamic pricing and advertising controls without concavity

Bertrand Crettez [a], Naila Hayek [a], Georges Zaccour [b,*]

[a] Université Panthéon-Assas, Paris II, CRED, France
[b] GERAD, HEC Montréal, Canada

ARTICLE INFO

Article history:
Received 14 September 2017
Received in revised form 8 January 2018
Accepted 8 January 2018

Keywords:
Advertising
Pricing
Goodwill stock
Nerlove–Arrow dynamics

ABSTRACT

We consider a pricing and advertising dynamic-optimization problem where the goodwill dynamics evolve à la Nerlove–Arrow. The firm maximizes its profit over a finite-planning horizon corresponding to the product's lifespan, and it turns out that the Hamiltonian is non-concave. We show the existence and uniqueness of an optimal solution under some mild conditions.

1. Introduction

An investment in advertising has a long-lasting effect, that is, it not only boosts the firm's demand and revenues today but also its future ones. The main reason for this is that advertising is a main input in building the brand's reputation, which in turn, is a significant driver of current and future sales. This simple observation in itself explains the existence of an extensive literature on *dynamic* advertising models and decisions, which started more than five decades ago and is still ongoing.

In this paper, we consider a continuous-time dynamic-optimization model à la Nerlove–Arrow and determine optimal pricing and advertising decisions. The cornerstone piece in a Nerlove–Arrow (N–A) model is the goodwill stock, a variable that positively affects sales. The firm can raise its goodwill stock, which is also referred to in the literature and in practice as brand reputation or brand equity, by investing in advertising, while part of this goodwill is lost as a result of consumers forgetting of the advertisement messages. In the parlance of dynamic optimization, the goodwill is a state variable that summarizes, in a compact way, the firm's current and past advertising outlays on its sales, which can also depend on other decision variables such as price and quality.

Our contribution to the Nerlove–Arrow class of advertising models is threefold. First, in any monopoly-profit-maximization model, one expects the price to be a decision variable, not a given parameter. Indeed, there is no valid conceptual reason to assume both a monopolistic environment and an exogenously given price during the whole planning horizon. By letting the price be a control variable, we add some realism to the stream of literature that only considered advertising. Second, as any product has a finite lifespan, we believe that the model must also have a finite terminal date. This reasoning especially holds when one assumes away any quality improvements over time for the product, which has been the norm rather than the exception in the dynamic advertising literature (see [13] and [42] for a discussion). The advantage of an infinite planning horizon resides in the fact that the optimal (or equilibrium in a competitive model) solution is stationary, which is typically easier to compute than a time-varying solution. Here, we stick to a finite horizon, and by the same token, provide insights into the firm's advertising and pricing trajectories in a more realistic context, and beyond the steady-state values that are often the focal point of the analysis in infinite-horizon models. The third contribution concerns the existence and uniqueness of an optimal solution. In our case, it turns out that the Hamiltonian function corresponding to the profit-maximization problem is not concave. Consequently, the usual sufficient conditions of optimality cannot be applied. We show, under some mild conditions, that the non-oscillating interior pricing and advertising solution is indeed optimal.

We shall refrain from extensively reviewing the literature and refer the reader to the comprehensive surveys in Feichtinger et al. [17] and Huang et al. [23]. (Other surveys of interest include [14,15,22,30].) Instead, we focus on the literature that is directly relevant to our paper, namely, the contributions that used an N–A

* Corresponding author.
E-mail addresses: bertrand.crettez@u-paris2.fr (B. Crettez), naila.hayek@u-paris2.fr (N. Hayek), georges.zaccour@gerad.ca (G. Zaccour).

https://doi.org/10.1016/j.orl.2018.01.004
0167-6377/

model, which is referred to as a capital stock advertising model in Feichtinger et al. [17], and highlight our contribution.

Table 1 provides an updated list of the papers covered in the survey in Huang et al. [23]. Each paper is characterized in terms of four features: (i) the price being or not being a decision variable; (ii) the planning horizon (finite or infinite); (iii) the type of strategic interactions; and finally (iv) a brief description of the main topic. A first observation based on Table 1 is that only 7 of the 36 listed papers included price as a decision variable, and in all these cases, the planning horizon was infinite. The conclusion here is that we do not know much about optimal pricing policies in the (probably more realistic) case of a finite terminal date. Note that the previous literature surveys in [43,17] and [14] included 19 papers using N–A dynamics, and only two of them had price as a decision variable, and both retained an infinite planning horizon. As price is clearly profit-relevant, we do believe that including it as a decision variable in a finite horizon setting fills an important gap in the literature. A second observation is that the N–A framework has been used in a wide variety of topics, both in an oligopoly/monopoly settings and in supply chains, which signals the framework's broad appeal. Finally, we note that 19 papers retained an infinite horizon, whereas 17 had a finite terminal date. In the later case, the focus has often been on the introduction of a new product or the management of advertising for a perishable (seasonal) product.

Our main results show that the optimal pricing policy follows the goodwill stock and is time-invariant. Further, the advertising trajectory is convex and monotone. Depending on the parameter values, advertising expenditures and the goodwill stock can be increasing or decreasing over time.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 presents the optimal non-oscillating interior solution, whose existence and uniqueness are shown in Section 4 under some mild conditions. Section 5 concludes.

2. Model

We consider a planning horizon $[0, T]$, with time t running continuously. The initial date corresponds to the introduction of a new product by the firm, and T to the end of the selling season. After T, the product loses its appeal because of, e.g., a change of season for fashion apparel, or the arrival of a new version for software. Denote by $p(t)$ the product's price at time $t \in [0, T]$, and by $G(t)$ the goodwill stock (brand reputation or brand equity). The demand function is given by

$$q(t) = \alpha G(t) - \beta p(t), \tag{1}$$

where α and β are strictly positive parameters. Following a long tradition in economics and management science, our demand function is affine, however with the additional feature that the market potential at any t is proportional to the brand goodwill.

Remark 1. Our demand function is micro-founded, i.e., it is derived from consumer's utility maximization problem. To show it, let the utility function of the representative consumer be given by the following quadratic function: $U(q, y) = \phi q - \frac{\kappa q^2}{2} + y$, where q is the demand for the firm's product, y is a composite good whose price is normalized to one, and ϕ and κ are positive parameters. The budget constraint is given by $pq + y = I$, where p is the price of the product and I the income. Maximizing $U(q, y)$ subject to the budget constraint yields the demand $q = \frac{\phi - p}{\kappa}$. It suffices to set $\alpha G = \frac{\phi}{\kappa}$ and $\beta = \frac{1}{\kappa}$ to get (1).

The goodwill stock evolves à la Nerlove–Arrow [40], i.e.,

$$\frac{dG}{dt}(t) = \dot{G}(t) = ka(t) - \delta G(t), \; G(0) = G_0 > 0, \tag{2}$$

where $a(t)$ is the advertising investment at time t, $k > 0$ is the marginal efficiency of advertising, and δ is the decay rate. Following a broad literature on dynamic advertising models (see, e.g., [29,17] and [23]), we assume that the advertising cost is convex increasing and given by the quadratic function $C(a) = \frac{\omega}{2} a^2(t)$, where ω is a positive parameter. Without any loss of generality, we suppose that the marginal production cost is constant and set equal to zero. An implication of this assumption is that the price can also be interpreted as a profit margin.

Assuming profit-optimizing behavior, the firm maximizes its stream of profits over the planning horizon, that is,

$$\max_{p(t),a(t)} = \int_0^T \left(p(t)\left(\alpha G(t) - \beta p(t)\right) - \frac{\omega}{2} a^2(t) \right) dt + s\, G(T), \tag{3}$$

subject to (2),

where $sG(T)$ is the salvage value of the brand at T, which measures the future profits that the firm can obtain from marketing products under the same brand name. We suppose that $S(G(T))$ can be approximated by a linear function, that is, $S(G(T)) = sG(T)$, where s is a positive parameter.

3. Optimal solution

We will highlight below that, since the optimization problem at hand is non-concave and the control set is unbounded, the existence and uniqueness of an optimal interior solution are far from assured. We will proceed in two steps. First, we present the optimal pricing and advertising decisions assuming the existence of an interior solution. Second, we provide a set of sufficient conditions that guarantees the existence and uniqueness of an optimal interior solution.

3.1. An interior solution

We start by making the following assumption, which will imply that the solution is not oscillating (i.e., the eigenvalues of the dynamical system are real numbers).

Assumption: The parameter values satisfy the inequality

$$\delta^2 - \frac{\alpha^2 k^2}{2\beta\omega} > 0. \tag{4}$$

Proposition 1. *Under condition (4), assume that there exists an interior solution; it is then given by*

$$p(t) = \frac{\alpha G(t)}{2\beta}, \tag{5}$$

$$a(t) = \frac{k\left(2\beta s\left((\delta + v)e^{v(T+t)} - (\delta - v)e^{v(T-t)}\right) + \alpha^2 G_0\left(e^{v(2T-t)} - e^{vt}\right)\right)}{2\beta\omega\left((\delta + v)e^{2vT} - (\delta - v)\right)}, \tag{6}$$

and the brand goodwill by

$$G(t) = \frac{2\beta s(\delta^2 - v^2)\left(e^{v(T+t)} - e^{v(T-t)}\right)}{\alpha^2\left((\delta + v)\,e^{2vT} - (\delta - v)\right)} - \frac{\alpha^2 G_0\left((\delta - v)e^{vt} - e^{v(2T-t)}(\delta + v)\right)}{\alpha^2\left((\delta + v)\,e^{2vT} - (\delta - v)\right)}, \tag{7}$$

where $v = \sqrt{\delta^2 - \frac{\alpha^2 k^2}{2\beta\omega}}$.

Proof. Introduce the Hamiltonian

$$H(p(t), a(t), G(t), \lambda(t)) = p(t)(\alpha G(t)) - \beta p(t) - \frac{\omega}{2} a^2(t) + \lambda(t)(ka(t) - \delta G(t)), \tag{8}$$