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Ruin probability in the dual risk model with two revenue streams

Esther Frostig

Department of Statistics, University of Haifa, Israel

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ABSTRACT

The dual risk model describes the surplus of a company with fixed expense rate and occasional random income inflows, called gains. Consider the dual risk model with two streams of gains. Type I gains arrive according to a Poisson process, and type II gains arrive according to a general renewal process. We show that the survival probability of the company can be expressed in terms of the survival probability in a dual risk process with renewal arrivals with initial reserve 0, and the survival probability in the dual risk process with Poisson arrivals in finite time.

1. Introduction

The dual risk model describes the surplus of a company with fixed expense rate, while gains arrive occasionally due to some contingent events (e.g. discoveries, sales). Thus, the surplus process $R(t)$ can be described as follows:

$$R(t) = u - ct + \mathcal{S}(t), \tag{1.1}$$

where u is the initial surplus, c is the constant rate at which expenses are paid out, $\mathcal{S}(t) = \sum_{i=1}^{N(t)} Y_i$; $N(t)$ is the number of revenue events (i.e. number of innovations, or discoveries) up to time t, Y_i are i.i.d, where Y_i is the income from the i gain event. $\{Y_i\}$ are independent of $N(t)$.

The dual risk model was introduced by Avanzi et al. [6] who considered the barrier dividend strategy for this model. We refer to their paper for applications where the dual model is appropriate. Since then, it has been studied for different dividends strategies, see e.g. [9] for similar model as in (1.1), [5,7] for the compound Poisson dual risk model perturbed by a Brownian motion, and [8,11,14,15] for a general spectrally positive Lévy processes.

We are motivated by the following scenario: Consider a company with a constant expense rate and two types of gain: Type I gain occurs due to random events like patents, discoveries, exit, selling some assets etc. Type II gain arrives periodically, due to dividend payments, rent etc. We are looking for the survival probability of the company, i.e. the probability that the surplus is always positive. The ruin probability is one minus the survival probability, i.e. the probability that the surplus will reach zero in a finite time.

In this note we consider two income arrival streams, type I and type II denoted by $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, respectively. Thus the surplus process $R(t)$ is described as follows:

$$R(t) = u - ct + \mathcal{S}_1(t) + \mathcal{S}_2(t), \tag{1.2}$$

where $\mathcal{S}_1(t) = \sum_{j=1}^{N_1(t)} Y_j$ and $\mathcal{S}_2(t) = \sum_{j=1}^{N_2(t)} Z_j$. N_1 is a Poisson process at rate λ, and Y_j is the income from the jth type I event. Y_j are i.i.d positive random variables distributed as Y, with distribution F_Y and Laplace transform $\tilde{y}(s) = \mathbb{E}[e^{-sY}]$. N_2 is a renewal process, where U_i is the inter-arrival time between the $i-1$ and the i arrival, $U_i, i \geq 2$ are i.i.d distributed as U, with distribution A, and the time until the first type II arrival is distributed as A_0. Z_j is the revenue of the jth type II event, $Z_j, j \geq 1$ are i.i.d positive random variables distributed as Z, with distribution function F_Z and Laplace transform $\tilde{z}(s) = \mathbb{E}[e^{-sZ}]$. N_1, N_2, $\{Y_j\}$, and $\{Z_j\}$ are independent.

In the sequel we will use some terminology from queueing theory, therefore we review some definitions. Consider a queueing system, where $N(t)$ is the number of customer arrivals up to time t, Y_i is the service time of the ith customer and u is the service time of the first customer that arrives to the system, let us assume at time 0. There is one server with service rate c. Assume that service starts at time 0 and continuous until the first time that the system is empty of customers. This period is called a busy period. It is the same as the time to ruin in the corresponding dual risk model (1.1). When u in (1.1) is replaced by a random variable Y_0, then if Y_0 has the same distribution as $Y_i, i \geq 1$, we call the corresponding busy period a regular busy period, otherwise we call it a special busy period where the service time of the customer that initiates the busy has different distribution than the other customers. The idle period starts when the busy period ends, and continuous until another busy period starts due to customer

E-mail address: frostig@stat.haifa.ac.il.

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arrival. We denote by $G/G/1$ a queueing system where the inter-arrival time are i.i.d with general distribution, the service times are i.i.d with general distribution, $c = 1$, and there is one server. $M/G/1$ queue is a special case of $G/G/1$ queue, where the inter-arrival are exponentially distributed, i.e. $N(t)$ is a Poisson process. For t less than the duration of the busy period, (1.1) describes the time to finish all the work in the corresponding queueing system at time t, i.e. the time it takes to finish servicing all the customers in the queue and the residual service time of the customer who is being served at this time. This quantity is called the workload or the virtual waiting-time at t.

Consider the surplus of the dual model as described by (1.1), where $N(t)$ is a general renewal process. The time to ruin coincides with the busy period in a G/G/1 queueing system with arrival process N and service times Y_i. Consider such a busy period that starts with workload Y_0 distributed as Y (i.e. a regular busy period). The probability that such a busy period is finite is given in formula (4.59) in [12]. It is the same as the ruin probability in the dual risk where the first claim arrives at time 0.

We are interested in the ruin probability for the model in (1.2), where ruin occurs the first time the surplus reaches 0. Related problems were studied in queueing theory. The most related paper is Ott [10]. He studied the virtual waiting time in a queueing system with two arrival streams as described above. He showed that in steady state, the virtual waiting time is distributed as the sum of the virtual waiting time in an $M/G/1$ queue, and the virtual waiting time in a $G/G/1$ queue, with inter-arrival time distributed as U, and service times distributed as the busy period in an $M/G/1$ queue, where the service time of the customer that initiates the busy period is distributed as Z and the service times of all other customers are distributed as Y. In our analysis we will apply a similar decomposition to obtain the survival probability, i.e. the probability that the surplus never reaches 0. In the context of queueing theory, Ott [10] assumed that the system is in steady state. However in the insurance risk models we usually assume that the net profit condition holds, this means that the system is not regenerative and the surplus goes to infinity with probability 1.

Throughout the paper, without loss of generality, we assume that $c = 1$ and we denote by G^{*n} the n-fold convolution of a distribution function G with itself.

2. Ruin probability

Let $\tau_0 = 0$, and let T_1 be the time of the first arrival from stream II. Let τ_1 be the first time after T_1, such that $R(\tau_1) = R(T_1^-)$, where $R(T_1^-)$ is the surplus just before the first type II arrival. τ_1 is defined as infinity if $R(t) > R(T_1^-)$ for all $t \geq T_1$. For $j \geq 1$, let T_j be the time of the first type II arrival after τ_{j-1}, (which is defined as ∞ if $\tau_{j-1} = \infty$), and let τ_j be the first time after T_j for which $R(\tau_j) = R(T_j^-)$, where $R(T_j^-)$ is the surplus just before T_j (defined as infinity if $R(t) > R(T_j^-)$ for all $t > T_j$), see Fig. 1. For $i \geq 1$, let $\mathcal{B}_i = \tau_i - T_i$ if $\tau_i < \infty$ and ∞ otherwise. Notice that $\mathcal{B}_i$ is the time to ruin in the dual risk model, where the first gain arrives at time 0, and it is type II gain. Consider the equivalent queueing system where the service regime is LIFO, i.e. the server always serves the customer that arrived last, and resumes the service of the customers that were already in the system after completing the service of the last arriving customer. This regime does not change the length of the busy period. Under this regime, when type II leaves the system there are no type I in the system that arrived after him. $\mathcal{B}_i$ defines a busy period of a queueing system that starts with type II arrival with service requirement Z, and ends when there are no type II customers and no type I customers that arrived after some of the type II customers. The duration of this busy period is the same as the duration of the busy period in the following $G/G/1$ queueing system: The inter-arrival time are i.i.d distributed as U, (as type II inter-arrival times between gain events in the original model). The service time is distributed as the busy period in an $M/G/1$ queue with arrival rate λ, the service times of all customers except the first one are i.i.d distributed as Y, and the service time of the first customer, that initiates the busy period, is distributed as Z. Let V be the generic service time and let $\tilde{v}(s) = \mathbb{E}[e^{-sV}]$ be its Laplace transform, then (see Prabhu [12], Chapter 2.2.):

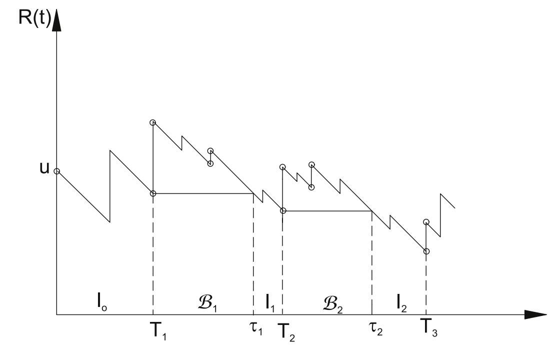

Fig. 1. The risk process: ∘ presents type II arrival.

$$\tilde{v}(s) = \tilde{z}(s + \lambda - \lambda\beta(s)), \tag{2.1}$$

where $\beta(s)$ is the Laplace transform of a regular busy period of an M/G/1 queue with arrival rate λ and service time distributed as Y, i.e.

$$\beta(s) = \tilde{y}(s + \lambda - \lambda\beta(s)). \tag{2.2}$$

Denote by B the distribution of V.

Let E_i be the event $\{\mathcal{B}_i = \infty\}$, and let $\bar{E}_i$ be its complement, i.e. $\bar{E}_i = \{\mathcal{B}_i < \infty\}$. Let

$$\psi = \mathbb{P}(E_i) = \mathbb{P}(\mathcal{B}_i = \infty).$$

Notice that ψ is the survival probability for the modified system with inter-gain arrivals distributed as U, i.i.d gains distributed as V, where the first gain arrives at time 0. In other words, ψ is the probability that a busy period in a $G/G/1$ queue with generic inter-arrival U, and generic service time V is infinity. ψ is given on p. 39–40 in [13], or in formula 4.59 in [12]:

$$\psi = \begin{cases} 0 & \text{if } \mathcal{A} = \infty \\ e^{-\mathcal{A}} & \text{if } \mathcal{A} < \infty, \end{cases} \tag{2.3}$$

where

$$\mathcal{A} = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} [1 - A^{*n}(t)] dB^{*n}(t).$$

Remark 2.1. Clearly, $\lambda\mathbb{E}[Y] > 1$ implies that $\mathbb{E}[V] = \infty$ and thus $\psi > 0$, and the ruin probability is less than 1. If $\lambda E[Y] < 1$,

$$\mathbb{E}[V] = \frac{\mathbb{E}[Z]}{1 - \lambda\mathbb{E}[Y]}.$$

From the general theory of random walk (Chapter 1 in [13]), $\psi = \mathbb{P}(\mathcal{B} = \infty) > 0$ iff

$$\frac{\mathbb{E}[Z]}{1 - \lambda\mathbb{E}[Y]} - \mathbb{E}[U] > 0$$

or

$$\frac{\mathbb{E}[Z]}{\mathbb{E}[U]} + \lambda\mathbb{E}[Y] > 1.$$