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Relative ageing of series and parallel systems: Effects of dependence and heterogeneity among components

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ABSTRACT

The effect of system structure on relative ageing properties of coherent systems has been extensively studied in terms of the increasing [reversed] hazard ratio. In this paper, we investigate the effects of dependence and heterogeneity among components on the relative ageing of series and parallel systems. Numerical examples are provided as illustrations.

1. Introduction

The notion of ageing, describing the variation of the performance of a unit with time, plays a crucial role in survival analysis and reliability theory. To this regard, various measures and measure-based stochastic orders have been developed to analyze ageing characteristics of life distributions. Two types of stochastic orders have been proposed in the literature to compare the relative ageing among units' lifetimes. The first type, known as transform orders, including the convex transform order, star order and superadditive order, is devoted to capturing the essence of how one distribution is more skewed than the other, and thus reflects that the latter distribution ages faster than the former. Moreover, several weaker orders are also proposed to compare skewness (relative ageing). These are DMRL order, NBUE order and HNBUE order, comparing the DMRL-ness, NBUE-ness and HNBUE-ness, respectively. A good discussion of this type of stochastic orders can be found in [1,8,10].

The second approach, reflecting the relative ageing by the increasing hazard ratio, was proposed and characterized by [6,20]. When the involved distributions have the same mean, such relative ageing order implies orderings of variances (see [11]). Two other notions of relative ageing are also defined by the increasing reversed hazard ratio (see [18]), and by increasing mean residual lifetime ratio (see [4]). In contrast to the first type of stochastic ordering, the second one has visible advantages since it has intuitive interpretation of relative ageing and can be used to model the phenomenon of crossing measures.

By employing the second type of relative ageing notions, [11] showed that the parallel system with additional redundant components ages faster in the sense of the increasing hazard ratio. Further, coherent systems with superior structures also turn out to ageing faster in terms of the (reversed) hazard ratio; see [2,16]. Recently, [12] investigated the effect of heterogeneity among components on the ageing speed of series and parallel systems. It was shown that the series and parallel systems with homogeneous PHR [PRHR] components age faster than the system with heterogeneous PHR [PRHR] components in terms of increasing reversed hazard [hazard] ratio. They also presented sufficient conditions for the relative ageing of parallel and series systems with scaled components.

It should be noted that all the works mentioned above only involve independent components. However, there is no study on the relative ageing among systems with dependent components. Motivated by this, the present paper is aiming at inquiring into the effects of dependence structure and heterogeneity among components on ageing properties of series and parallel systems. Archimedean copula is a useful tool to model dependence behaviors, and has been commonly used in the literature to capture the dependence among components in k-out-of-n systems; see [13,19]. With the help of Archimedean copulas, we show that the series [parallel] system comprised of two types of independent scaled components becomes age faster in terms of the hazard rate

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[reversed hazard rate] by increasing their heterogeneity and the number of components with better performance.

The rest of the paper is structured as follows. Section 2 recalls pertinent definitions and notions used in the sequel. Section 3 investigates effects of dependence structure among components and the size of system on the ageing speed of series and parallel systems with identical components. Section 4 studies the effect of heterogeneity among components on the relative ageing of parallel and series systems comprising independent components.

2. Preliminaries

Throughout this article, we assume that all random variables are continuous and defined on $\mathbb{R}_+ = [0, +\infty)$. Let $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$, where $\mathbb{N}$ denotes the natural number set. The inverse functions defined in this paper are assumed to be right continuous. In what follows, '$a \overset{\text{sign}}{=} b$' means a and b have the same sign. For a differentiable function f, we use f', f'' and $f^{(k)}$ to represent its first, second and kth derivatives.

Let X_1 and X_2 be two non-negative random variables with respective density functions f_1 and f_2, distribution functions F_1 and F_2, survival functions $\bar{F}_1$ and $\bar{F}_2$, respectively. Let $h_1(x) = f_1(x)/\bar{F}_1(x)$ and $h_2(x) = f_2(x)/\bar{F}_2(x)$ be the hazard rate functions of X_1 and X_2, $\tilde{r}_1(x) = f_1(x)/F_1(x)$ and $\tilde{r}_2(x) = f_2(x)/F_2(x)$ be the reversed hazard rate functions of X_1 and X_2, respectively.

Definition 2.1. X_1 is said to age faster than X_2 in the

(i) hazard rate (denoted by $X_1 \lesssim_{\text{hr}} X_2$) if $h_1(x)/h_2(x)$ is increasing $x \in \mathbb{R}_+$;
(ii) reversed hazard rate (denoted by $X_1 \lesssim_{\text{rh}} X_2$) if $\tilde{r}_2(x)/\tilde{r}_1(x)$ is increasing $x \in \mathbb{R}_+$.

Let $x_{1:n} \le \cdots \le x_{n:n}$ be the ordered components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$. For two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, $\boldsymbol{x}$ is said to *majorize* $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$ and $\sum_{i=1}^j x_{i:n} \le \sum_{i=1}^j y_{i:n}$ for $j = 1, \ldots, n-1$; $\boldsymbol{x}$ is said to *weakly majorize* $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\text{w}}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^j x_{i:n} \le \sum_{i=1}^j y_{i:n}$ for $j = 1, \ldots, n$. It is worth mentioning that the majorization order implies the weak majorization order, while the reverse is not true in general. The notion of majorization is useful in establishing various inequalities arising from reliability theory and actuarial science. For more details, one may refer to [14].

Archimedean copulas have been widely used in actuarial science, reliability theory and many other areas due to its mathematical tractability and the capability of capturing wide ranges of dependence. By definition, for a decreasing and continuous function $\phi : \mathbb{R}_+ \mapsto [0, 1]$ such that $\phi(0) = 1$ and $\phi(+\infty) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse,

$$C_\phi(u_1, \ldots, u_n) = \phi\left(\psi(u_1) + \cdots + \psi(u_n)\right), \quad \text{for all } u_i \in [0, 1],\ i = 1, 2, \ldots, n,$$

is called an *Archimedean copula* with the generator ϕ if $(-1)^k\phi^{(k)}(x) \ge 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex. It is common knowledge that the Archimedean family contains a great many useful copulas, including the well-known independence (product) copula, the Clayton copula and the Ali–Mikhail–Haq (AMH) copula. For more discussions on copulas and their properties, one may refer to [15,17].

3. Dependent homogeneous components

In this section, we assume all concerned systems consist of dependent homogeneous components and such dependent structure is captured by Archimedean copula.

Theorem 3.1. *Suppose two sets of homogeneous random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ have common Archimedean copula with generator ϕ.*

(i) *If $t\ln'\left[-\phi'(t)/(1-\phi(t))\right]$ is decreasing in $t \in \mathbb{R}_+$, then $Y_{m:m} \lesssim_{\text{hr}} X_{n:n}$, for any $m \ge n$.*
(ii) *If $t\ln'\left[-\phi'(t)/\phi(t)\right]$ is decreasing (increasing) in $t \in \mathbb{R}_+$, then $X_{n:n} \lesssim_{\text{rh}} (\gtrsim_{\text{rh}}) Y_{m:m}$, for any $m \ge n$.*

Proof. (i) The distribution functions of $X_{n:n}$ and $Y_{m:m}$ can be written as

$$F_{X_{n:n}}(x) = \phi\left(n\psi(F(x))\right) \text{ and } F_{Y_{m:m}}(x) = \phi\left(m\psi(F(x))\right),\ x \in \mathbb{R}_+.$$

Then, the hazard rate functions of $X_{n:n}$ and $Y_{m:m}$ are computed as

$$h_{X_{n:n}}(x) = \frac{nf(x)\psi'(F(x))\phi'\left(n\psi(F(x))\right)}{1-\phi\left(n\psi(F(x))\right)} \quad \text{and}$$
$$h_{Y_{m:m}}(x) = \frac{mf(x)\psi'(F(x))\phi'\left(m\psi(F(x))\right)}{1-\phi\left(m\psi(F(x))\right)}.$$

Let $u = F(x)$. It suffices to prove that

$$\Delta_1(u) = \frac{h_{Y_{m:m}}(x)}{h_{X_{n:n}}(x)} = \frac{m}{n} \times \frac{\phi'\left(m\psi(u)\right)}{1-\phi\left(m\psi(u)\right)} \times \left[\frac{\phi'\left(n\psi(u)\right)}{1-\phi\left(n\psi(u)\right)}\right]^{-1}$$

is increasing in $u \in [0, 1]$. Taking the derivative of $\Delta_1(u)$ with respective to u gives rise to

$$\begin{aligned}
\Delta_1'(u) &\overset{\text{sign}}{=} \left[\frac{\phi'\left(m\psi(u)\right)}{1-\phi\left(m\psi(u)\right)}\right]' \times \frac{\phi'\left(n\psi(u)\right)}{1-\phi\left(n\psi(u)\right)} - \left[\frac{\phi'\left(n\psi(u)\right)}{1-\phi\left(n\psi(u)\right)}\right]' \times \frac{\phi'\left(m\psi(u)\right)}{1-\phi\left(m\psi(u)\right)} \\
&\overset{\text{sign}}{=} \frac{\psi'(u)[1-\phi\left(m\psi(u)\right)]}{\psi(u)\phi'(m\psi(u))} \times \left[\frac{m\psi(u)\phi''(m\psi(u))}{1-\phi\left(m\psi(u)\right)} + \frac{m\psi(u)\phi'^2(m\psi(u))}{[1-\phi\left(m\psi(u)\right)]^2}\right] \\
&\quad - \frac{\psi'(u)[1-\phi\left(n\psi(u)\right)]}{\psi(u)\phi'(n\psi(u))} \times \left[\frac{n\psi(u)\phi''(n\psi(u))}{1-\phi\left(n\psi(u)\right)} + \frac{n\psi(u)\phi'^2(n\psi(u))}{[1-\phi\left(n\psi(u)\right)]^2}\right] \\
&\overset{\text{sign}}{=} \left[\frac{n\psi(u)\phi''(n\psi(u))}{\phi'(n\psi(u))} + \frac{n\psi(u)\phi'(n\psi(u))}{1-\phi(n\psi(u))}\right] - \left[\frac{m\psi(u)\phi''(m\psi(u))}{\phi'(m\psi(u))} + \frac{m\psi(u)\phi'(m\psi(u))}{1-\phi(m\psi(u))}\right] \\
&= t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]\Bigg|_{t=n\psi(u)} - t\ln'\left[-\frac{\phi'(t)}{1-\phi(t)}\right]\Bigg|_{t=m\psi(u)} \\
&\ge 0,
\end{aligned}$$

where the inequality is due to the assumption that $t\ln'\left[-\phi'(t)/(1-\phi(t))\right]$ is decreasing in $t \in \mathbb{R}_+$. Hence, the desired result is obtained.

(ii) According to (i), the reversed hazard rate functions of $X_{n:n}$ and $Y_{m:m}$ are given by

$$\tilde{r}_{X_{n:n}}(x) = \frac{nf(x)\psi'(F(x))\phi'\left(n\psi(F(x))\right)}{\phi\left(n\psi(F(x))\right)} \quad \text{and}$$
$$\tilde{r}_{Y_{m:m}}(x) = \frac{mf(x)\psi'(F(x))\phi'\left(m\psi(F(x))\right)}{\phi\left(m\psi(F(x))\right)}.$$

To demonstrate (ii), it suffices to show

$$\Delta_2(u) = \frac{\tilde{r}_{Y_{m:m}}(F^{-1}(u))}{\tilde{r}_{X_{n:n}}(F^{-1}(u))} = \frac{m}{n} \times \frac{\phi'\left(m\psi(u)\right)}{\phi\left(m\psi(u)\right)} \times \left[\frac{\phi'\left(n\psi(u)\right)}{\phi\left(n\psi(u)\right)}\right]^{-1}.$$