



On the sample path properties of mixed Poisson processes

Miaoqi Fu, Xianhua Peng*

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong



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ABSTRACT

The mixed Poisson process has been widely used in financial engineering for modeling arrival of events that cluster in time, as it has strictly stationary and positively correlated increments. However, we show that, surprisingly, the sample autocovariance and autocorrelation of the increments of a mixed Poisson process converge to zero almost surely as the sample size goes to infinity. Consequently, the sample autocovariance or autocorrelation cannot be used in the method of moments for parameter estimation of mixed Poisson processes.

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1. Introduction

The mixed Poisson process is a generalization of the homogeneous Poisson process for modeling the occurrence of events. In contrast to the homogeneous Poisson process that has independent increments, the mixed Poisson process has the distinctive feature that the correlation between its increments is positive. Roughly speaking, if the number of events occurring in the current period is large, then the number of events that will occur in the next period will also tend to be large. Hence, the arrival of events under the mixed Poisson process has the feature of *clustering in time*, which has been empirically observed in various fields such as finance and insurance.

The mixed Poisson process has been widely used in the literature of financial engineering and insurance. The CreditRisk+ system [19] uses a mixed Poisson process with the Gamma distribution as the mixing distribution for the modeling of arrival of default events. [6,7] develop an importance sampling method for evaluating the loss distribution under a mixed Poisson model of portfolio credit risk. [4] provides a review of jump processes such as mixed Poisson processes and doubly stochastic Poisson processes and their applications in credit risk modeling. [12] proposes a model for pricing portfolio credit derivatives in which a mixed Poisson process is used as a market factor that can introduce clustering shocks to affect the default probabilities of firms. [10] proposes a mixed Poisson credit risk model in which loss-given-default is dependent on probability of default. [2] provides comprehensive discussion on credit risk models including the CreditRisk+

and its extensions. The mixed Poisson process has also been widely used for modeling the arrival of accidents, sickness, and insurance claims in the actuarial science literature (see, e.g., [13,20]).

In this paper, we study the asymptotics of the sample autocovariance and autocorrelation of the increments of a mixed Poisson process. Contrasting the homogeneous Poisson process, the increments of a mixed Poisson process are strictly stationary but have positive autocorrelation. Hence, one may expect that the asymptotic distribution of sample autocorrelation of the increments of a mixed Poisson process has a mean equal to the true positive autocorrelation (as the sample size goes to infinity), which may hold for a weakly stationary time series under some general conditions. For example, [3, Chap. 7] shows that the asymptotic mean of sample autocorrelation is equal to the population autocorrelation for weakly stationary ARMA(p, q) time series and for weakly stationary time series $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, where $\{Z_t\}$ is a sequence of i.i.d. random variables with mean zero and finite variance. See also [5, Chap. 6].

The main result of this paper is that, surprisingly, the sample autocovariance and autocorrelation of the increments of a mixed Poisson process converge to zero almost surely. This provides an interesting counterexample showing that even for a strictly stationary time series, the sample autocorrelation may not have an asymptotic mean that is equal to the population autocorrelation.

The main result of the paper also implies that the sample autocovariance or autocorrelation cannot be used in the method of moments for parameter estimation of a mixed Poisson process. The method of moments based on autocovariance is a commonly used method for parameter estimation for weakly stationary time series partly due to the simplicity of the method. For example, the parameters of AR(p) model can be consistently estimated by

* Corresponding author.

E-mail addresses: miaof2fmq@gmail.com (M. Fu), maxhpeng@ust.hk (X. Peng).

the Yule–Walker equation, which is based on equating population autocorrelation with sample autocorrelation (see, e.g., [3, Chap. 8.1]). Although we have not noticed any existing literature that use the method of moments for the estimation of parameters for mixed Poisson processes, our result provides a precaution against the use of such a method for mixed Poisson processes.

The paper also complements the study of the properties of mixed Poisson process in the literature of point processes and applied probability. [1] discusses the goodness-of-fit test of whether a point process can be adequately modeled as a mixed Poisson process and in particular a homogeneous Poisson process based on the characterization of a mixed Poisson process. [11] shows that the property that normalized event occurrence times are the order statistics of independent uniform random variables on (0, 1) characterizes the mixed Poisson processes within the class of general point processes. The characterization of mixed Poisson processes within the class of birth processes, stationary point processes, and general point processes are also studied in [1,9,13,14,17], among others; see Chapter 6 of [8] for comprehensive discussion and summary. When the mixing distribution of the mixed Poisson process has a gamma distribution, the mixed Poisson process is called a Pólya–Lundberg process (or Pólya process); see [16] for a comprehensive discussion. [15] provides a martingale characterization of Pólya–Lundberg processes within the class of mixed Poisson processes.

The remainder of the paper is organized as follows. In Section 2, we briefly review the properties of the increments of mixed Poisson processes. In Section 3, we prove the main result of this paper. Section 4 concludes.

2. The increments of mixed Poisson processes

We recall the definition of the mixed Poisson process (see, e.g., [18, Section 8.5.1]). A counting process $\{N(t), t \geq 0\}$ is called a mixed Poisson process if there exists a positive random variable Λ with distribution function $F_\Lambda(\cdot)$ such that for any n , any sequence $\{k_i \mid i = 1, 2, \dots, n\}$ of nonnegative integers, and any sequence $0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_n \leq b_n$,

$$P(N(b_i) - N(a_i) = k_i; i = 1, 2, \dots, n) = \int_0^\infty \prod_{i=1}^n \frac{(\lambda(b_i - a_i))^{k_i}}{k_i!} e^{-\lambda(b_i - a_i)} dF_\Lambda(\lambda). \tag{1}$$

If $\{N(t), t \geq 0\}$ is a mixed Poisson process, then for any fixed t , the marginal distribution of $N(t)$ is a mixed Poisson distribution. Comparing with Poisson distribution, mixed Poisson distribution is overspersed in the sense that the variance of $N(t)$ is larger than $E[N(t)]$. A mixed Poisson process is a non-homogeneous pure birth process whose birth rate function is determined by the mixing distribution (see, e.g., [18, Theorem 8.5.1]). The interoccurrence times of a mixed Poisson process are identically distributed but not independent.

The easiest way to construct a mixed Poisson process is through stochastic time change of a Poisson process. More precisely, let Λ be a random variable with distribution function $F_\Lambda(\cdot)$, $\{M(t), t \geq 0\}$ be a Poisson process with intensity 1, and let Λ and $\{M(t), t \geq 0\}$ be independent. Define the process $\{N(t), t \geq 0\}$ as follows:

$$N(t) := M(\Lambda \cdot t), t \geq 0. \tag{2}$$

Then, the process $\{N(t), t \geq 0\}$ defined in Eq. (2) is a mixed Poisson process (see, e.g., Definition 4.3 in [8, p. 66]).

Let $\delta > 0$ be a constant and consider the increments of the mixed Poisson processes:

$$m_t := N(t\delta) - N((t-1)\delta), t = 1, 2, \dots \tag{3}$$

It is clear from (1) that the time series m_t is strictly stationary but not independent. In fact, the autocovariance (ACV) and autocorrelation (ACR) functions of m_t at any lag are positive, as is shown in the following proposition.

Proposition 2.1. $\{m_t \mid t = 1, 2, \dots\}$ defined in (3) is a strictly stationary time series with the following autocovariance (ACV) and autocorrelation (ACR) functions:

$$c_k = Cov(m_t, m_{t-k}) = \delta^2 Var(\Lambda) > 0, \forall k > 0, \tag{4}$$

$$\rho_k = \frac{Cov(m_t, m_{t-k})}{Var(m_t)} = \frac{\delta^2 Var(\Lambda)}{\delta E[\Lambda] + \delta^2 Var(\Lambda)} > 0, \forall k > 0. \tag{5}$$

Proof. By direct computation. \square

In particular, if $N(t)$ is a Pólya–Lundberg process with the Gamma distribution with shape parameter α and scale parameter β as the mixing distribution, then the ACV and ACR of $\{m_t\}$ are given by

$$c_k = Cov(m_t, m_{t-k}) = \delta^2 \alpha \beta^2, \rho_k = \frac{Cov(m_t, m_{t-k})}{Var(m_t)} = \frac{\delta \beta}{1 + \delta \beta}, \forall k > 0.$$

3. Main result

The main result of this paper is that the sample ACV and sample ACR at any lag of $\{m_t\}$ converge to zero almost surely, although the population ACV and population ACR of $\{m_t\}$ are strictly positive. As a result, the sample ACV or sample ACR cannot be used in the method of moments for parameter estimation of mixed Poisson processes. The sample ACV and sample ACR at lag k of $\{m_t\}$ are defined as follows:

$$\bar{m}_T := \frac{1}{T} \sum_{t=1}^T m_t = \frac{1}{T} N(T\delta) \tag{6}$$

$$\hat{c}_{k,T} := \frac{\sum_{t=k+1}^T (m_t - \bar{m}_T)(m_{t-k} - \bar{m}_T)}{T}. \tag{7}$$

$$\hat{\rho}_{k,T} := \frac{\sum_{t=k+1}^T (m_t - \bar{m}_T)(m_{t-k} - \bar{m}_T)}{\sum_{t=1}^T (m_t - \bar{m}_T)^2}. \tag{8}$$

First, we show that the sample ACV at any lag of the increments of a homogeneous Poisson process converge to zero almost surely.

Lemma 3.1. Let $M(t)$ be a homogeneous Poisson process with intensity 1. Let $\lambda > 0$ be a constant and $k > 0$ be a constant integer. Define

$$m_{t,\lambda} := M(t\delta\lambda) - M((t-1)\delta\lambda), \tag{9}$$

$$\bar{m}_{T,\lambda} := \frac{1}{T} \sum_{t=1}^T m_{t,\lambda}, \tag{10}$$

$$\hat{c}_{k,T}(\lambda) := \frac{\sum_{t=k+1}^T [m_{t,\lambda} - \bar{m}_{T,\lambda}][m_{t-k,\lambda} - \bar{m}_{T,\lambda}]}{T}. \tag{11}$$

Then, it holds that

$$\lim_{T \rightarrow \infty} \hat{c}_{k,T}(\lambda) = 0, \text{ a.s.} \tag{12}$$

Proof. Omitted due to limited space. Details can be provided upon request. \square

Second, we consider the mixed Poisson process $N(t)$ defined in (2). For this process $N(t)$, an intuitive but unrigorous argument to

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