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Sharp probability bounds for the binomial moment problem with symmetry

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ABSTRACT

Sharp bounds in terms of the first few binomial moments are found for the probability of a union of events, when the random variable denoting the number of events that occur follows symmetric distribution. Connection between the bounds of this paper and the bounds from a special case of the binomial moment problem of Prekopa (1995) is shown. As a special case, bounds for the probability when the underlying probability distribution is unimodal-symmetric are also found.

1. Introduction

Formulating a linear programming problem to find the sharp bounds for the probability of a union of events, for which the first few moments are assumed to be known, is studied in detail in [13] as moment problem. Many other variations of this problem, such as finding the sharp bounds for the probability of occurrence of at least r, exactly r events are studied in [2,6]. Bounds for the probability of the union of events are also found from disaggregated linear programming problems [14,18]. In [3] bounds for the probability of the union of events are found by restricting the variables of the dual linear programming problem to three different polyhedral regions. The bounds from these regions are compared with one another to find the best of all. In [11] modified inclusion–exclusion formula is found to efficiently calculate the probability of the union of events in n-space. In [12] the dual feasible bases structures and the bounds are obtained for the discrete moment problem with fractional moments. Even though complete knowledge of the probability distribution is not assumed, assuming just the shape of the probability distribution, the bounds are obtained in [9,15,16]. Other than the technique of linear programming, many other techniques are also employed in order to find the sharp bounds for the probability of the union of events in terms of some of its binomial or power moments [1,4,5,8,10]. Using graph theory technique, the bounds are obtained for the probability of the union of events in [7,17].

In this paper, sharp bounds are found for the probability of the union of events when the underlying distribution is symmetric using an LPP formulation. The outline of the paper is as follows. In Section 2, the linear programming problem is formulated. In Section 3, the structures of the dual feasible bases are given. In Section 4, the bounds presented in this paper are proved to coincide with the bounds from the binomial moment problem formulated in [13], for a particular collection of the binomial moments. In Sections 5–7, the bounds are given for the cases where different set of binomial moments are assumed. In Section 8, as a special case, the bounds are given for the probability of the union of events when the underlying probability distribution is unimodal-symmetric. In Section 9, numerical examples are discussed.

2. Problem formulation

Let $A_1, A_2, \ldots, A_n$ be $n \ (\geq 3)$ arbitrary events in an arbitrary probability space and γ be the random variable that denotes the number of events that occur out of these n events. Let $p_i = P\left(\gamma = i\right), \ i = 0, 1, \ldots, n$. The first few binomial moments of the probability distribution are assumed to be known.

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If kth binomial moment of the probability distribution is denoted by S_k, then the problem of finding bounds for the probability of the union of n events when the first q binomial moments are given is mathematically formulated in [13] as follows:

$$min\,(max) \sum_{i=1}^{n} p_i$$

subject to

$$\sum_{i=0}^{n} \binom{i}{k} p_i = S_k, k = 1, \dots, q \tag{1}$$

$$p_i \geq 0, i = 0, \dots, n.$$

Now the probability distribution is assumed to be symmetric, i.e., $p_i = p_{n-i}, \;\; i = 0, 1, \dots, \frac{n-1}{2}$, when n is odd and $p_i = p_{n-i}, \;\; i = 0, 1, \dots, \frac{n}{2}$, when n is even. Since for any symmetric distribution, all of the odd central moments are zero and any kth central moment can be written as a linear combination of the first k binomial moments for any probability distribution, the set $\{S_1, S_2, S_3, \dots, S_n\}$ is linearly dependent (where $S_1, S_2, S_3, \dots, S_n$ correspond to a symmetric distribution). However the subset $\{S_1, S_2, S_4, S_6, \dots\}$ is linearly independent. Hence, only the first odd binomial moment S_1 together with some of the even binomial moments is considered. Then the problem (1) can be rewritten as follows:

$$min\,(max)\, p_0 + \sum_{i=1}^{\frac{n-1}{2}} 2p_i$$

subject to

$$\sum_{i=0}^{\frac{n-1}{2}} \left(\binom{i}{k} + \binom{n-i}{k} \right) p_i = S_k, k = 1, 2, 4, 6, \dots, 2\,(m-1)$$
$$\left(m \leq \frac{n+1}{2} \right) \tag{2}$$

$$p_i \geq 0, i = 0, \dots, \frac{n-1}{2}$$

when n is odd and as

$$min(max) p_0 + \sum_{i=1}^{\frac{n}{2}-1} 2p_i + p_{\frac{n}{2}}$$

subject to

$$\sum_{i=0}^{\frac{n}{2}-1} \left(\binom{i}{k} + \binom{n-i}{k} \right) p_i + \binom{\frac{n}{2}}{k} p_{\frac{n}{2}} = S_k, k = 1, 2, 4, 6, \dots, 2\,(m-1)$$
$$\left(m \leq \frac{n}{2} + 1 \right) \tag{3}$$

$$p_i \geq 0, i = 0, \dots, \frac{n}{2}$$

when n is even.

3. Structures of dual feasible bases

Let the matrix A, whose ith column is denoted by a_{i-1}, be the coefficient matrix of the problems (2) and (3), B be an $m \times m$ part of the coefficient matrix A and I_B be the subscript set of those columns of A that form the basis B. The cost vector is denoted by C and C_B denotes the $m \times 1$ part of C that includes those elements of C that are basic.

Theorem 3.1. *For the problems* (2) and (3), *any dual feasible basis, in terms of the subscripts, has one of the following structures.*
(*i*) *For problem* (2):

	m is even	m is odd
min problem	$\left\{0, i, i+1, \dots, j, j+1, \frac{n-1}{2}\right\}$	$\{0, i, i+1, \dots, j, j+1\}$
max problem	$\{0, 1, i, i+1, \dots, j, j+1\}$ or $I_B \subset \left\{1, \dots, \frac{n-1}{2}\right\}$	$\left\{0, 1, i, i+1, \dots, j, j+1, \frac{n-1}{2}\right\}$ or $I_B \subset \left\{1, \dots, \frac{n-1}{2}\right\}$

All dual feasible bases are dual non-degenerate, except for those bases with $I_B \subset \left\{1, \dots, \frac{n-1}{2}\right\}$ *which are dual degenerate.*
(*ii*) *For problem* (3):