Accepted Manuscript

A note on representations of linear inequalities in non-convex mixed-integer quadratic programs

Adam N. Letchford, Daniel J. Grainger

PII: S0167-6377(17)30386-3
DOI: https://doi.org/10.1016/j.orl.2017.10.007
Reference: OPERES 6285

To appear in: *Operations Research Letters*

Received date : 11 July 2017
Revised date : 26 September 2017
Accepted date : 11 October 2017

Please cite this article as: A.N. Letchford, D.J. Grainger, A note on representations of linear inequalities in non-convex mixed-integer quadratic programs, *Operations Research Letters* (2017), https://doi.org/10.1016/j.orl.2017.10.007

A Note on Representations of Linear Inequalities in Non-Convex Mixed-Integer Quadratic Programs

Adam N. Letchford

Department of Management Science, Lancaster University, Lancaster LA1 4YW, UK.

Daniel J. Grainger

Former PhD student at Lancaster University.

Abstract

In the literature on the quadratic 0-1 knapsack problem, several alternative ways have been given to represent the knapsack constraint in the quadratic space. We extend this work by constructing analogous representations for arbitrary linear inequalities for arbitrary non-convex mixed-integer quadratic programs with bounded variables.

Keywords: mixed-integer nonlinear programming, global optimisation, semidefinite programming

1. Introduction

It has long been known that one can obtain useful reformulations of $\mathcal{NP}$-hard optimisation problems by introducing additional variables representing products or squares of original variables. This idea has been applied to, e.g., 0-1 linear programs [18, 22], 0-1 quadratic programs [1, 11, 19], non-convex quadratic programs [7, 23], non-convex quadratically constrained quadratic programs [12, 21], and many other problems.

The presence of additional variables leads to some flexibility in the choice of representing linear constraints. In their paper on semidefinite programming relaxations of the 0-1 quadratic knapsack problem (QKP), Helmberg *et al.* [16] consider three different representations of the knapsack constraint. In his thesis, Helmberg [14], considers also a fourth representation. The purpose of this note is to extend this work, by constructing analogous representations for arbitrary linear inequalities for arbitrary mixed-integer quadratic programs (MIQPs). The only restriction is that all variables involved in the inequality must be explicitly bounded.

The paper has a very simple structure. The literature is reviewed in Section 2, and the new results are in Section 3.

Email addresses: a.n.letchford@lancaster.ac.uk (Adam N. Letchford), grainged@gmail.com (Daniel J. Grainger)

2. Literature Review

We now give a brief overview of the relevant literature. We cover general quadratic 0-1 programs in Subsection 2.1, the QKP in Subsection 2.2, and bounded MIQP in Subsection 2.3.

2.1. Quadratic 0-1 programming

A *quadratic 0-1 program* (0-1 QP) with n variables and m constraints is a problem of the form

$$\max\left\{x^T Qx : \; Ax \le b, \; x \in \{0,1\}^n\right\},$$

where $Q = \{q_{ij}\} \in \mathbb{Q}^{n\times n}$, $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. It is well known that 0-1 QPs are strongly $\mathcal{NP}$-hard, yet have many important applications (see, e.g., [4, 10]).

One can convert 0-1 QPs into 0-1 linear programs using a standard linearisation trick, due to Fortet [11]. We introduce for all $1 \le i < j \le n$ the binary variable x_{ij}, representing the product $x_i x_j$. To simplify notation, we identify x_{ji} with x_{ij}. The 0-1 QP is then formulated as:

$$\begin{array}{lcl}
\max & \sum_{1\le i\le n} q_{ii}x_i + \sum_{1\le i<j\le n}(q_{ij}+q_{ji})x_{ij} & \\
\text{s.t.} & Ax \le b & \\
 & x_{ij} \le x_i & (1 \le i \le n,\ j \ne i) \\
 & x_i + x_j \le x_{ij} + 1 & (1 \le i < j \le n) \\
 & x_i \in \{0,1\}^n & (1 \le i \le n) \\
 & x_{ij} \in \{0,1\}^n & (1 \le i < j \le n).
\end{array}$$