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The joint distribution of the sample minimum and maximum from a smooth distribution on $[w_1, w_2]$

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A R T I C L E I N F O

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We give expansions in inverse powers of the sample size *n* for the joint distribution function of the sample extremes when sampling from any smooth distribution on a finite interval when its density function is positive at its end points.

1. Introduction

Many variables in real life take values in finite intervals. The best known examples are proportions. Often interest is on extreme values of such variables, for example, lowest and highest proportions of people affected by a deadly disease.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution on a finite interval. Suppose Z_{n1} denotes a suitably normalized min $(X_1, X_2, \ldots, X_n)$ and belongs to the Weibull domain of attraction such that

$$Prob.\,(Z_{n1} > z_1) \to e^{-z_1} \tag{1}$$

as $n \to \infty$. Suppose Z_{n2} denotes a suitably normalized -max $(X_1, X_2, \ldots, X_n)$ and belongs to the Weibull domain of attraction such that

$$Prob.\,(Z_{n2} > z_2) \to e^{-z_2} \tag{2}$$

as $n \to \infty$. Both (1) and (2) hold if for example the sample is from the uniform [0, 1] distribution.

There is much work on the asymptotic joint distribution of sample minimum and sample maximum. For example, Leadbetter, Lindgren, and Rootzen (1983) showed under certain general conditions that sample minimum and sample maximum are asymptotically independent. If (1) and (2) hold then under certain general conditions

$$Prob.\,(Z_{n1} > z_1, Z_{n2} > z_2) \to e^{-z_1-z_2}, \tag{3}$$

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} Prob.\,(Z_{n1} \le z_1, Z_{n2} \le z_2) \to e^{-z_1-z_2} \tag{4}$$

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and

$$EZ_{n1}^r Z_{n2}^s \to r!s! \tag{5}$$

as $n \to \infty$. In practice, n is finite. It can never be infinity. So, models based on (3)–(5) cannot be realistic. Realistic models can be obtained by deriving expansions for the joint distribution of the minimum and maximum. Liao and Peng (2015) and Lu and Peng (2017) give the most recent work on the asymptotics of the joint distribution of the minimum and maximum.

The aim of this note is to derive expansions for the distributions of the minimum and maximum of a random sample of values in a finite interval. We derive expansions for the joint distribution function, joint density function and product moments. These expansions can be used for improved statistical modeling of the minimum and maximum. The expansions for the joint distribution function can be used for improved percentile estimation. For instance, by Corollary 3.1,

$$Prob.\left(Z_{n1} > z_1, Z_{n2} > z_2\right) = e^{-z_1-z_2}\left(1 + Q_1 n^{-1}\right)$$

which can be expected to perform better than (1). The expansions for the joint density function can be used for improved maximum likelihood estimation. For instance, by Corollary 3.1,

$$\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} Prob.\left(Z_{n1} \le z_1, Z_{n2} \le z_2\right) = e^{-z_1-z_2}\left(1 + q_1' n^{-1}\right)$$

which can be expected to perform better than (2). The expansions for the moments can be used for improved moments estimation. For instance, by Corollary 3.1,

$$EZ_{n1}^r Z_{n2}^s = r!s! + M_{rs1} n^{-1}$$

which can be expected to perform better than (3).

The known work giving expansions for the joint distribution of sample minimum and sample maximum has focused on specific distributions or processes: a Brownian motion (Choi & Roh, 2013); the normal distribution (Shiryaeva, 2013); autoregressive process of order one (Withers & Nadarajah, 2015); and so on. We are aware of no work giving expansions for the joint distribution for general classes.

Let $X_1, \ldots, X_n$ be a random sample with minimum m_n and maximum M_n from any distribution function $F(x)$ on a finite interval $[w_1, w_2]$ with finite derivatives $\{F_{\cdot r}(x),\ r \ge 0\}$ at its endpoints w_1, w_2. However, $F_{\cdot r}(x)$ need not exist in (w_1, w_2).

Section 2 gives the joint distribution function and product moments of

$$\mathbf{Y}_n = \left(Y_{n1}, Y_{n2}\right) = n\left(m_n - w_1, w_2 - M_n\right) \tag{6}$$

in terms of

$$A_r = F_{\cdot r}\left(w_1\right)/r!,\ B_r = F_{\cdot r}\left(w_2\right)/r! \tag{7}$$

as expansions in powers of n^{-1}. Note that

$$Y_{n1} \ge 0,\ Y_{n2} \ge 0,\ Y_{n1} + Y_{n2} \le e_n = n\left(w_2 - w_1\right), \tag{8}$$

and $B_0 = 1, A_1 \ge 0, B_1 \ge 0$. We assume that $A_0 = 0$ (that is F does not have an atom at w_1), and $A_1 > 0, B_1 > 0$. So, the density function of $F(x)$ is positive at its end points. For example, this rules out the beta distribution except for the uniform distribution.

Section 3 gives the joint distribution function and product moments of

$$\mathbf{Z}_n = \left(Z_{n1}, Z_{n2}\right) = \left(A_1 Y_{n1}, B_1 Y_{n2}\right) = n\left(A_1\left(m_n - w_1\right),\ B_1\left(w_2 - M_n\right)\right) \tag{9}$$

in terms of

$$a_r = \overline{a}_r = A_r/A_1^r,\ b_r = B_r/B_1^r,\ \overline{b}_r = (-1)^r b_r \tag{10}$$

as expansions in powers of n^{-1} under the same conditions. This considerably simplifies the results of Section 2. Sections 4 and 5 give the distribution function and moments of the marginals of $\mathbf{Y}_n$ and $\mathbf{Z}_n$ under the same conditions. Section 6 shows that $\mathrm{corr}(Z_{n1}, Z_{n2}) = O\left(n^{-1}\right)$ and outlines how to deal with the case $A_1 = 0$ or $B_1 = 0$.

We need some notations. For $n, r = 0, 1, \ldots,$ set $(n)_r = n!/(n-r)! = n(n-1)\cdots(n-r+1)$. Let $\delta_{i,j} = I(i=j)$ denote the Kronecker delta function. For a sequence $\mathbf{c} = (c_1, c_2, \ldots)$ of real or complex numbers, define the partial ordinary Bell polynomial, $\widetilde{B}_{r,k} = \widetilde{B}_{r,k}(\mathbf{c})$, by

$$\left(\sum_{r=1}^{\infty} c_r t^r\right)^k = \sum_{r=k}^{\infty} \widetilde{B}_{r,k} t^r$$

for $t \in C$, the plane of complex numbers, and $k = 0, 1, \ldots$. Comtet (1974) tabled $\widetilde{B}_{r,k}$ on page 309 for $1 \le r \le 10$. Bell polynomials are valuable tools for handling power series.