ARTICLE IN PRESS
Journal of the Korean Statistical Society ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of the Korean Statistical Society

journal homepage: www.elsevier.com/locate/jkss

Modified proportional hazard rates and proportional reversed hazard rates models via Marshall–Olkin distribution and some stochastic comparisons

Narayanaswamy Balakrishnan [a], Ghobad Barmalzan [b,*], Abedin Haidari [c]

[a] Department of Mathematics and Statistics, McMaster University, Hamilton, Canada
[b] Department of Statistics, University of Zabol, Sistan and Baluchestan, Iran
[c] Department of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran

ARTICLE INFO

Article history:
Received 2 June 2017
Accepted 18 October 2017
Available online xxxx

AMS 2000 subject classifications:
60E15
90B25

Keywords:
Modified proportional hazard rates model
Modified proportional reversed hazard rates model
Proportional hazard rates model
Proportional reversed hazard rates model
Weak majorization order
P-larger order
Order statistics
Usual stochastic order
Marshall–Olkin distribution

ABSTRACT

Adding parameters to a known distribution is a useful way of constructing flexible families of distributions. Marshall and Olkin (1997) introduced a general method of adding a shape parameter to a family of distributions. In this paper, based on the Marshall–Olkin extension of a specified distribution, we introduce two new models, referred to as modified proportional hazard rates (MPHR) and modified proportional reversed hazard rates (MPRHR) models, which include as special cases the well-known proportional hazard rates and proportional reversed hazard rates models, respectively. Next, when two sets of random variables follow either the MPHR or the MPRHR model, we establish some stochastic comparisons between the corresponding order statistics based on majorization theory. The results established here extend some well-known results in the literature.

1. Introduction

In practical situations, observed lifetime data often have monotone, bathtub or upside-down bathtub shapes in its hazard rate. In such cases, a distribution should have considerable flexibility to reflect some of these characteristics and shapes. For this purpose, two methods can be used. The first method is to use some well-known families of distributions such as gamma, Weibull and log-normal, which have been studied quite extensively in the literature; see, for example, Johnson, Kotz, and Balakrishnan (1994, 1995). The second method is to expand a family of distributions by adding a parameter for more flexibility. This method has been used considerably; for example, the family of Weibull distributions is constructed by taking powers of exponentially distributed random variables. We shall now recall a known procedure for this purpose. Suppose G is a baseline distribution function with support $\mathbb{R}^+$ and corresponding survival function $\bar{G}$. Consider the following distributions:

$$F(x;\alpha) = \frac{G(x)}{1-\bar{\alpha}\,\bar{G}(x)}, \qquad x,\alpha \in \mathbb{R}^+,\ \bar{\alpha} = 1-\alpha, \tag{1.1}$$

* Corresponding author.
E-mail address: ghobad.barmalzan@gmail.com (G. Barmalzan).

https://doi.org/10.1016/j.jkss.2017.10.003
1226-3192/

$$F(x;\alpha)=\frac{\alpha\, G(x)}{1-\bar{\alpha}\, G(x)},\qquad x,\alpha\in\mathbb{R}^+,\ \bar{\alpha}=1-\alpha. \tag{1.2}$$

Clearly, (1.1) and (1.2) are both valid cumulative distribution functions. Moreover, we may note that if α in (1.1) is changed to $1/\alpha$, we obtain the family of distributions in (1.2). Marshall and Olkin (1997) originally proposed the family of distributions in (1.1) and studied it for the case when G is a Weibull distribution. When G has probability density and hazard rate functions as g and r_G, respectively, then the hazard rate function corresponding to F in (1.1) is given by

$$r_F(x;\alpha)=\frac{1}{1-\bar{\alpha}\,\bar{G}(x)}\,r_G(x),\qquad x,\ \alpha\in\mathbb{R}^+,\ \bar{\alpha}=1-\alpha.$$

Therefore, one can observe that if $r_G(x)$ is decreasing (increasing) in x, then for $0<\alpha\le 1\,(\alpha\ge 1)$, $r_F(x;\alpha)$ is also decreasing (increasing) in x. Moreover, one can observe that $r_G(x)\le r_F(x;\alpha)$ for $0<\alpha\le 1$, and $r_F(x;\alpha)\le r_G(x)$ for $\alpha\ge 1$. For this reason, the parameter α in (1.1) is referred to as a tilt parameter (see Marshall and Olkin (2007, p. 242)). Thus, an interesting property of this extended form of distribution is that it has a flexible hazard function depending on the value of the added parameter α, which renders this new distribution to be more applicable than the baseline distribution G in many practical situations. For example, let us use the exponential distribution with $\bar{G}(x)=e^{-\lambda x}$ in (1.1). We then find

$$F(x;\alpha,\lambda)=\frac{1-e^{-\lambda x}}{1-\bar{\alpha}\,e^{-\lambda x}},\qquad x,\alpha,\lambda\in\mathbb{R}^+,\bar{\alpha}=1-\alpha. \tag{1.3}$$

The family of distributions in (1.3) is called the extended exponential distribution with shape parameter α and scale parameter λ (denoted by $EE(\alpha,\lambda)$) in the literature. It is of interest to note that while the exponential distribution has a constant hazard rate, the extended exponential distribution in (1.3) has decreasing hazard rate for $\alpha\le 1$ and increasing hazard rate for $\alpha\ge 1$ (see Marshall and Olkin (1997)). The extension in (1.1) has been used recently to extend several known distributions; for example, Cordeiro and Lemonte (2012), Gupta, Lvin, and Peng (2010), Hirose (2002), and Marshall and Olkin (1997) studied the family of distributions in (1.1) when G is either exponential or Weibull distribution. Some other special cases of (1.1) that have been studied in the literature have considered G to be Pareto (Ghitany, 2005), gamma (Ristic, Jose, & Ancy, 2007), Lomax (Ghitany, Al-Awadhi, & Al-khalfan, 2007), linear failure-rate (Ghitany & Kotz, 2007), q-Weibull (Jose, Naik, & Ristic, 2008) and Birnbaum–Saunders (Lemonte, 2013) distributions.

It is worthwhile to note that the distribution functions in (1.1) and (1.2) can be viewed as the distribution functions of geometric random minima. More precisely, suppose $X_1,X_2,\ldots$ is a sequence of independent and identically distributed random variables with common distribution function G. Further, let M and N be two random variables which are independent of the X_i's, with respective probability mass functions $P(M=m)=\alpha\,(1-\alpha)^{m-1}$ for $\alpha\le 1$, and $P(N=n)=\frac{1}{\alpha}\,(1-\frac{1}{\alpha})^{n-1}$ for $\alpha>1$, where $m,n=1,2,\ldots$. Now, set $Y_{1:M}=\min(X_1,X_2,\ldots,X_M)$ and $Z_{1:N}=\min(X_1,X_2,\ldots,X_N)$. Then, it can be verified that the distribution function of $Y_{1:M}$ and $Z_{1:N}$ are precisely the distribution functions in (1.1) and (1.2), respectively. By using a similar approach, Aly and Benkherouf (2011) restated this problem and studied the distributional properties of $Y_{1:M}$ when M has a probability mass function defined on $\mathbb{N}=\{1,2,\ldots\}$. Interested readers may refer to Kirmani and Gupta (2001) for a comprehensive discussion on this topic. On the other hand, from (1.1), we readily observe

$$\frac{\bar{F}(x;\alpha)}{F(x;\alpha)}=\alpha\,\frac{\bar{G}(x)}{G(x)},\qquad x,\alpha\in\mathbb{R}^+. \tag{1.4}$$

The relation in (1.4) is the so-called proportional odds ratio model which was introduced by Bennett (1983) and used in the analysis of a lung cancer trial. The close connection between the proportional odds ratio model and the family of distributions in (1.1) has been discussed in detail by Kirmani and Gupta (2001) and Sankaran and Jayakumar (2006). Some ordering results in the proportional odds ratio model have been discussed by Gupta and Peng (2009). Further, Li and Zhao (2011) and Nanda and Das (2011) have considered a mixture version of the odds ratio model when the tilt parameter itself is a random variable and established some ordering results and aging properties.

The proportional reversed hazard rates (PRHR) and proportional hazard rates (PHR) models are two flexible families of distributions which have been considered extensively in reliability and survival analysis. Suppose $X_1,\ldots,X_n$ denote the independent lifetimes of n components of a system with survival functions $\bar{F}_1,\ldots,\bar{F}_n$ and distribution functions $F_1,\ldots,F_n$, respectively. Then, $X_1,\ldots,X_n$ are said to follow the PHR model if there exist positive constants $\lambda_1,\ldots,\lambda_n$ and a baseline survival function $\bar{F}(x)$ such that $\bar{F}_i(x)=\bar{F}^{\lambda_i}(x)$ for $i=1,\ldots,n$. Similarly, $X_1,\ldots,X_n$ are said to follow the PRHR model if there exist positive constants $\alpha_1,\ldots,\alpha_n$ and a baseline distribution function $F(x)$ such that $F_i(x)=F^{\alpha_i}(x)$ for $i=1,\ldots,n$. For additional discussion about the PHR and PRHR models, one may refer to Chapter 7 of Marshall and Olkin (2007).

In the following, we utilize the PHR (PRHR) model as baseline model in (1.1) ((1.2)) to define two new models, referred to as modified proportional hazard rates (MPHR) and modified proportional reversed hazard rates (MPRHR) models.

Definition 1.1. Suppose $X_1,\ldots,X_n$ are independent lifetimes of n components of a system with respective survival functions $\bar{F}_1,\ldots,\bar{F}_n$. Then, $X_1,\ldots,X_n$ are said to follow the MPHR model with tilt parameter α, modified proportional hazard rates