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Least squares estimator of fractional Ornstein–Uhlenbeck processes with periodic mean

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ABSTRACT

We first study the drift parameter estimation of the fractional Ornstein–Uhlenbeck process (fOU) with periodic mean for every $\frac{1}{2} < H < 1$. More precisely, we extend the consistency proved in Dehling et al. (2016) for $\frac{1}{2} < H < \frac{3}{4}$ to the strong consistency for any $\frac{1}{2} < H < 1$ on the one hand, and on the other, we also discuss the asymptotic normality given in Dehling et al. (2016). In the second main part of the paper, we study the strong consistency and the asymptotic normality of the fOU of the second kind with periodic mean for any $\frac{1}{2} < H < 1$.

1. Introduction

Consider the fractional Ornstein–Uhlenbeck process (fOU) $X = \{X_t, t \geqslant 0\}$ given by the following linear stochastic differential equation

$$dX_t = -\alpha X_t dt + dB_t^H, \quad X_0 = 0, \tag{1}$$

where α is an unknown parameter, and $B^H = \left\{B_t^H, t \geqslant 0\right\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$.

The drift parameter estimation problem for the fOU X observed in continuous time and discrete time has been studied by using several approaches (see Kleptsyna and Le Breton (2002); Hu and Nualart (2010); Hu and Song (2013); Brouste and Iacus (2012); El Onsy, Es-Sebaiy, and Viens (2017); Es-Sebaiy and Ndiaye (2014)). In a general case when the process X is driven by a Gaussian process, El Machkouri, Es-Sebaiy, and Ouknine (2016) studied the non-ergodic case corresponding to $\alpha < 0$. They provided sufficient conditions, based on the properties of the driving Gaussian process, to ensure that least squares estimators-type of α are strongly consistent and asymptotically Cauchy. On the other hand, using Malliavin-calculus advances (see Nourdin and Peccati (2012)), Es-Sebaiy and Viens (2016) provided new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and they used their result to study drift parameter estimation problems for some stochastic differential equations driven by fractional Brownian motion with fixed-time-step observations (in particular for the fOU X given in (1) with $\alpha > 0$). Similarly, in Sottinen and Viitasaari (2017) the authors studied an ergodicity estimator for the parameter α in (1), where the fractional Brownian motion is replaced with a general Gaussian process having stationary increments.

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Recently, Dehling, Franke, and Woerner (2016) studied a drift parameter estimation problem for the above equation (1) with slight modifications on the drift. More precisely, they considered the following fractional Ornstein–Uhlenbeck process with periodic mean function

$$dX_t = \left(\sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X_t\right) dt + dB_t^H, \quad X_0 = 0 \tag{2}$$

where B^H is a fBm with Hurst parameter $\frac{1}{2} < H < 1$, the functions $\varphi_i, i = 1, \ldots, p$ are bounded by a constant $C > 0$ and periodic with the same period $\nu > 0$, and the real numbers $\mu_i, i = 1, \ldots, p$ together with $\alpha > 0$ are considered unknown parameters. The motivation comes from the fact that such equation can be used to model time series which are a combination of a stationary process and periodicities. Indeed, in order to model different phenomena with stationary time series, one often have to take periodic behaviour into account (for comprehensive study of time series analysis in discrete time, we refer to Hamilton (1994)). In Dehling et al. (2016) the authors proposed the least squares estimator (LSE) to estimate $\theta := (\mu_1, \ldots, \mu_p, \alpha)^\top$ based on the continuous-time observations $\{X_t, 0 \leqslant t \leqslant n\nu\}$ as $n \to \infty$. For the sake of simplicity, we assume that the functions $\varphi_i, i = 1, \ldots, p$ are orthonormal in $L^2([0, \nu], \nu^{-1}dt)$, i.e. $\int_0^\nu \varphi_i(t)\varphi_j(t)\nu^{-1}dt = \delta_{ij}$. We also choose $\nu = 1$. Finally, throughout the paper we denote by symbol δ, e.g. if $dX_t = u_t dt + dB_t^H$, $\int_0^n X_t \delta X_t := \int_0^n X_t u_t dt + \int_0^n X_t \delta B_t^H$, the Skorokhod integral with respect to fBm B^H (see Appendix for definition) while d stands for pathwise integral.

Let us consider the LSE $\widehat{\theta}_n$ of θ given in Dehling et al. (2016) by

$$\widehat{\theta}_n := Q_n^{-1} P_n \tag{3}$$

where

$$P_n := \left(\int_0^n \varphi_1(t)dX_t, \ldots, \int_0^n \varphi_p(t)dX_t, -\int_0^n X_t \delta X_t\right)^\top, \quad Q_n = \begin{pmatrix} G_n & -a_n \\ -a_n^\top & b_n \end{pmatrix}$$

with

$$G_n := \left(\int_0^n \varphi_i(t)\varphi_j(t)dt\right)_{1\leqslant i,j\leqslant p};$$

$$a_n^\top := \left(\int_0^n \varphi_1(t)X_t dt, \ldots, \int_0^n \varphi_p(t)X_t dt\right); \quad b_n := \int_0^n X_t^2 dt.$$

Let us describe what is known about the asymptotic behaviour of $\widehat{\theta}_n$: if $\frac{1}{2} < H < \frac{3}{4}$, then

- as $n \to \infty$,

$$\widehat{\theta}_n \longrightarrow \theta, \tag{4}$$

  in probability, see Dehling et al. (2016, Theorem 1);
- as $n \to \infty$,

$$n^{1-H}\left(\widehat{\theta}_n - \theta\right) \text{ converges in law to a normal distribution,} \tag{5}$$

  see Dehling et al. (2016, Theorem 2).

In the first part of our paper we extend the convergence in probability (4) proved when $\frac{1}{2} < H < \frac{3}{4}$ to the almost sure convergence for every $\frac{1}{2} < H < 1$. More precisely, we establish the strong consistency for the LSE $\widehat{\theta}_n$ for every $\frac{1}{2} < H < 1$. At the same time we correct the covariance matrix of the normal limit distribution given in Dehling et al. (2016, Theorem 1) which is established based on a flawed technique.

Our second main interest in this paper is to estimate the drift parameters of the fractional Ornstein–Uhlenbeck process of the second kind with periodic mean, that is the solution of the following equation

$$dX_t^{(1)} = \left(\sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X_t^{(1)}\right) dt + dY_t^{(1)}, \ X_0 = 0 \tag{6}$$

where $Y_t^{(1)} := \int_0^t e^{-s} dB_{a_s}^H$ with $a_t = He^{\frac{t}{H}}$ and B^H is a fBm with Hurst parameter $\frac{1}{2} < H < 1$. The fractional Ornstein–Uhlenbeck process of the second kind without periodicities have some interesting features. Firstly, it possess short range dependence even for $H > \frac{1}{2}$ which makes it interesting model for many purposes. Secondly, it is closely related to the Lamperti transformation of the fractional Brownian motion. The parameter estimation for the fOU of the second kind without periodicities is well studied in several recent papers (see Es-Sebaiy and Viens (2016); Azmoodeh and Morlanes (2013); Azmoodeh and Viitasaari (2015); El Onsy, Es-Sebaiy, and Tudor (in press)). In comparison to the fractional Ornstein–Uhlenbeck process of the first kind, one obtains normal limiting distribution for any value of the Hurst parameter $H > \frac{1}{2}$.