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Hierarchical Archimax copulas

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ABSTRACT

The class of Archimax copulas is generalized to hierarchical Archimax copulas in two ways. First, a hierarchical construction of d-norm generators is introduced to construct hierarchical stable tail dependence functions which induce a hierarchical structure on Archimax copulas. Second, by itself or additionally, hierarchical frailties are introduced to extend Archimax copulas to hierarchical Archimax copulas in a similar way as nested Archimedean copulas extend Archimedean copulas. Possible extensions to nested Archimax copulas are discussed. A general formula for the density and its evaluation of Archimax copulas is also introduced.

1. Introduction

The class of Archimax copulas [5,7] generalizes Archimedean copulas to incorporate a stable tail dependence function as known from extreme-value copulas. Archimax copulas thus include both Archimedean and extreme-value copulas as special cases. They provide a link between dependence structures arising in multivariate extremes and Archimedean copulas, which have intuitive and computationally appealing properties. One feature of Archimedean copulas is that they can be nested in the sense that under assumptions detailed below, one can plug Archimedean copulas into each other and still obtain a proper copula. Such a construction is hierarchical in the sense that certain multivariate margins are exchangeable, yet the copula overall is not; this additional flexibility to allow for (partial) asymmetry over an exchangeable model is typically used to model components belonging to different groups, clusters or business sectors.

In this work, we raise the following natural question, addressed in Sections 2 and 3: How can hierarchical Archimax copulas be constructed? Since we work with stochastic representations, sampling is also covered. Constructing nested Archimax copulas is largely an open problem which we discuss in Appendix A. Moreover, to fill a gap in the literature, we present a general formula for the density and its evaluation of Archimax copulas; see Appendix B. In what follows, we assume the reader to be familiar with the basics of Archimedean copulas (ACs) and extreme-value copulas (EVCs); see, e.g., [32] for the former (from which we also adopt the notation) and Chapter 6 of [24] for the latter.

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2. Hierarchical extreme-value copulas via hierarchical stable tail dependence functions

2.1. Connection between d-norms and stable tail dependence functions

A copula C is an extreme-value copula if and only if it is max-stable, i.e., if $C(\boldsymbol{u}) = C^m(u_1^{1/m}, \dots, u_d^{1/m})$ for all $m \in \mathbb{N}$ and $\boldsymbol{u} = (u_1, \dots, u_d) \in [0, 1]^d$; see, e.g., Theorem 6.2.1 in [24]. An extreme-value copula C can be characterized in terms of its stable tail dependence function $\ell : [0, \infty)^d \to [0, \infty)$ via the relation

$$C(\boldsymbol{u}) = \exp\{-\ell(-\ln u_1, \dots, -\ln u_d)\} \tag{1}$$

valid for all $\boldsymbol{u} \in [0, 1]^d$; see, e.g., Section 8.2 in [2] and Chapter 6 in [24]. A characterization of stable tail dependence functions ℓ (being homogeneous of order 1, being 1 when evaluated at the unit vectors in $\mathbb{R}^d$ and being fully d-max decreasing) is given in [7] and [39].

Sampling from EVCs is often quite challenging and time consuming. Examples which are comparably easy to sample are Gumbel and nested Gumbel copulas, which are the only Archimedean and nested Archimedean EVCs, respectively; in both cases, a stochastic representation is available, as reported in [12] and Theorem 4.5.2 in [34]. Specifically, the Gumbel (or logistic) copula C with parameter $\alpha \in (0, 1]$ and stable tail dependence function defined, for all $\boldsymbol{x} \in [0, \infty)^d$, by $\ell(\boldsymbol{x}) = (x_1^{1/\alpha} + \cdots + x_d^{1/\alpha})^\alpha$ can be sampled using the algorithm of [28]. It exploits the stochastic representation

$$\boldsymbol{U} = (\psi(E_1/V), \dots, \psi(E_d/V)) \sim C, \tag{2}$$

where $\psi(t) = e^{-t^\alpha}$ is a Gumbel generator, $E_1, \dots, E_d$ are mutually independent $\mathcal{E}(1)$, also independent of the frailty $V \sim \mathcal{PS}(\alpha) = \mathcal{S}\{\alpha, 1, \cos^{1/\alpha}(\alpha\pi/2), \mathbf{1}_{(\alpha=1)}; 1\}$; see p. 8 of [36] for the parameterization of this α-stable distribution. As for nested Gumbel copulas [42], they can be sampled based on a stochastic representation corresponding to the nesting structure; see [30]. The main idea is to replace the single frailty V by a sequence of dependent frailties (all α-stable for different α), nested in a specific way; see Section 3.

For more complicated EVCs, approximate or exact simulation schemes have been proposed in [8,9,40] based on the following stochastic representation of max-stable processes; see [13,38,40].

Theorem 1 (*Spectral Representation of Max-Stable Processes*)**.** *Let $W_1(\boldsymbol{s}), W_2(\boldsymbol{s}), \dots$ be mutually independent copies of the random process $W(\boldsymbol{s})$, $\boldsymbol{s} \in \mathcal{S} \subseteq \mathbb{R}^q$, such that $W(\boldsymbol{s}) \geq 0$ and $\mathrm{E}\{W(\boldsymbol{s})\} = 1$ for all $\boldsymbol{s} \in \mathcal{S}$. Furthermore, let $P_1, P_2, \dots$ be points of a Poisson point process on $[0, \infty)$ with intensity $x^{-2}\,\mathrm{d}x$. Then*

$$Z(\boldsymbol{s}) = \sup_{i\in\mathbb{N}}\{P_i W_i(\boldsymbol{s})\} \tag{3}$$

is a max-stable random process with unit Fréchet margins and, for all $x_1, \dots, x_d > 0$,

$$\ell(x_1, \dots, x_d) = \mathrm{E}\left[\max_{j\in\{1,\dots,d\}}\{x_j W(\boldsymbol{s}_j)\}\right] \tag{4}$$

is the associated stable tail dependence function of the random vector $(Z(\boldsymbol{s}_1), \dots, Z(\boldsymbol{s}_d))$ for fixed $\boldsymbol{s}_1, \dots, \boldsymbol{s}_d$. Therefore, if a process $Z(\boldsymbol{s})$ can be expressed as in (3), the distribution function of the random vector $(Z(\boldsymbol{s}_1), \dots, Z(\boldsymbol{s}_d))$ is $\Pr\{Z(\boldsymbol{s}_1) \leq x_1, \dots, Z(\boldsymbol{s}_d) \leq x_d\} = \exp\{-\ell(1/x_1, \dots, 1/x_d)\}$, i.e., $(Z(\boldsymbol{s}_1), \dots, Z(\boldsymbol{s}_d))$ has EVC C with stable tail dependence function ℓ and unit Fréchet margins $\exp(-1/x_j)$ for all $j \in \{1, \dots, d\}$.

For completeness, Algorithm 1 describes the traditional approach for simulating max-stable processes constructed using (3). This algorithm goes back to [40] and provides approximate simulations by truncating the supremum to a finite number of processes in (3). When the random process $W(\boldsymbol{s})$ is bounded almost surely, a stopping criterion may be designed to optimally select the number of Poisson points N to perform exact simulation. For more general exact sampling schemes, we refer to [8] and [9].

Algorithm 1. An approximate sampling of max-stable processes based on (3) proceeds as follows:

1. Simulate Poisson points $P_1, \dots, P_N$ in decreasing order as $P_i = 1/(E_1 + \cdots + E_i)$, $i \in \{1, \dots, N\}$, where $E_1, \dots, E_N \sim \varepsilon(1)$ are mutually independent standard exponential random variables.
2. Simulate N mutually independent copies $W_1(\boldsymbol{s}), \dots, W_N(\boldsymbol{s})$ of the process $W(\boldsymbol{s})$ at a finite set of locations $\boldsymbol{s} \in \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_d\}$.
3. For each location $\boldsymbol{s} \in \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_d\}$, set $Z(\boldsymbol{s}) = \max\{P_1 W_1(\boldsymbol{s}), \dots, P_N W_N(\boldsymbol{s})\}$.

By choosing the spatial domain $\mathcal{S}$ in (3) to be finite and replacing $W(\boldsymbol{s}_1), \dots, W(\boldsymbol{s}_d)$ by non-negative random variables $W_1, \dots, W_d$ with $\mathrm{E}(W_j) = 1$ for all $j \in \{1, \dots, d\}$, thus replacing the random process $W(\boldsymbol{s})$ by the non-negative random vector $\boldsymbol{W} = (W_1, \dots, W_d)$, this representation also provides a characterization of, and sampling algorithms for, (finite-dimensional) EVCs; from here on we will adopt this "vector case" for W and accordingly for Z.