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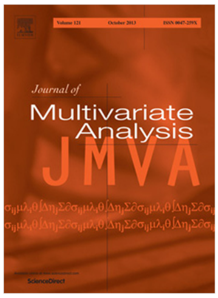

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Efron's monotonicity property for measures on $\mathbb{R}^2$

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Abstract

In this paper, we prove some kernel representations for the covariance of two functions taken on the same random variable and we deduce kernel representations for some functionals of a continuous one-dimensional measure. Then we apply these formulas to extend Efron's monotonicity property, given in Efron [14] and valid for independent log-concave measures, to the case of general measures on $\mathbb{R}^2$. The new formulas are also used to derive some further quantitative estimates in Efron's monotonicity property.

Keywords: covariance identity, log-concave, functional inequality, monotonicity, measurement error
2010 MSC: 60E15, 62E10

1. Introduction: A monotonicity property

Efron [14] proved the following result.

Proposition 1. *Let (X, Y) be a pair of real-valued random variables. Then the following statements are equivalent:*

(i) *For any $\Psi : \mathbb{R}^2 \to \mathbb{R}$, a function which is nondecreasing in each argument, the conditional expectation*

$$I(s) = \mathrm{E}\{\Psi(X, Y) | X + Y = s\} \tag{1}$$

is nondecreasing in s.

(ii) *For any $(x, y) \in \mathbb{R}^2$, the conditional survival functions*

$$S_X(x; s) = \Pr(X > x | X + Y = s) \quad \text{and} \quad S_Y(y; s) = \Pr(Y > y | X + Y = s) \tag{2}$$

are nondecreasing in s.

In this paper, condition (i) of Proposition 1 is referred to as Efron's "monotonicity property". Efron [14] used Proposition 1 to prove the monotonicity property for independent log-concave variables X and Y. In this paper, we extend the validity of Efron's monotonicity property to more general pairs (X, Y) on the plane; see Section 4. Our main result, Theorem 1, provides a condition on the joint density h of (X, Y), in terms of the second derivatives of $\varphi = -\ln h$ which implies (ii) of Proposition 1. In particular, in Section 4.3 we exhibit examples of random pairs satisfying the monotonicity property that are neither log-concave nor mutually independent. We also recover by different techniques Efron's monotonicity for independent log-concave variables in Section 4.2. Then we obtain quantitative lower-bounds for the derivative of Efron's I function in Section 5.

Our proofs rely on several key covariance identities which are stated in Section 3. These identities, originating in Hoeffding [21] (see also [22] for a translation of the German original), build on more recent results in the log-concave case due to Menz and Otto [35]. We conclude the paper in Section 6 by providing complete proofs of the key covariance identities stated in Section 3 together with some further examples and counterexamples.

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