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Abstract

The orientation of a rigid object can be described by a rotation that transforms it into a standard position. For a symmetrical object the rotation is known only up to multiplication by an element of the symmetry group. Such ambiguous rotations arise in biomechanics, crystallography and seismology. We develop methods for analyzing data of this form. A test of uniformity is given. Parametric models for ambiguous rotations are presented, tests of location are considered, and a regression model is proposed. An example involving orientations of diopside crystals (which have symmetry of order 2) is used throughout to illustrate how our methods can be applied.

Keywords: Frame, Orientation, Regression, Symmetric array, Symmetry, Test of location, Test of uniformity
2010 MSC: 62H11, 62H15, 62H20

1. Introduction

Data that are rotations of $\mathbb{R}^3$ occur in various areas of science, such as palaeo-magnetism [25, 36, 46], plate tectonics and seismology [2, 16, 24, 44, 47], biomechanics [28, 39, 43], crystallography [15, 18], and texture analysis, i.e., analysis of orientations of crystalites [11, 26, 27]. The sample space is the 3-dimensional rotation group, $SO(3)$, and methods for handling such data are now an established part of directional statistics; see Mardia & Jupp [30, Section 13.2]. In some contexts the presence of symmetry means that the rotations are observed subject to ambiguity, so that it is not possible to distinguish a rotation $\mathbf{X}$ from $\mathbf{XR}$ for any rotation $\mathbf{R}$ in some given subgroup K of $SO(3)$. From the mathematical point of view, the sample space is the quotient $SO(3)/K$ of $SO(3)$ by K. Such spaces arise in many scientific contexts: the case in which K is generated by the rotations through 180° about the coordinate axes gives the orthogonal axial frames considered by Arnold & Jupp [1], which can be used to describe aspects of earthquakes; several groups K of low order occur as the symmetry groups of crystals; the icosahedral group is the symmetry group of some carborane molecules [20], of most closed-shell viruses [17], of the natural quasicrystal, icosahedrite [4], and of the blue phases of some liquid crystals [42, Section 6.1.2].

There are two very natural approaches to statistics on $SO(3)/K$:

(i) the *averaging approach*, in which each point $[\mathbf{X}] = \{\mathbf{XR} : \mathbf{R} \in K\}$ of $SO(3)/K$ is regarded as the 'average' of the $|K|$ points $\mathbf{XR}$, where $\mathbf{R}$ runs through K and $|K|$ denotes the order of K,

(ii) the *embedding approach*, which uses a function $\mathbf{t} : SO(3)/K \to E$ to transform elements $[\mathbf{X}]$ of $SO(3)/K$ into vectors $\mathbf{t}([\mathbf{X}])$ in some inner-product space E.

A typical use of the averaging approach is the averaging of a probability density f on $SO(3)$ to give a corresponding probability density $\bar{f}$ on $SO(3)/K$, defined by

$$\bar{f}([\mathbf{X}]) = |K|^{-1} \sum_{\mathbf{R}\in K} f(\mathbf{XR}). \qquad (1)$$

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