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Asymptotic normality of quadratic forms with random vectors of increasing dimension

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ABSTRACT

This paper provides sufficient conditions for the asymptotic normality of quadratic forms of averages of random vectors of increasing dimension and improves on conditions found in the literature. Such results are needed in applications of Owen's empirical likelihood when the number of constraints is allowed to grow with the sample size. Indeed, the results of this paper are already used in Peng and Schick (2013) for this purpose. We also demonstrate how our results can be used to obtain the asymptotic distribution of the empirical likelihood with an increasing number of constraints under contiguous alternatives. In addition, we discuss potential applications of our result. The first example focuses on a chi-square test with an increasing number of cells. The second example treats testing for the equality of the marginal distributions of a bivariate random vector. The third example generalizes a result of Schott (2005) by showing that a standardized version of his test for diagonality of the dispersion matrix of a normal random vector is asymptotically standard normal even if the dimension increases faster than the sample size. Schott's result requires the dimension and the sample size to be of the same order.

1. Introduction

Let r_n be positive integers that tend to infinity with n. Let $\xi_{n,1},\dots,\xi_{n,n}$ be independent and identically distributed r_n-dimensional random vectors with mean $\mathrm{E}(\xi_{n,1})=0$ and dispersion matrix $V_n=\mathrm{E}(\xi_{n,1}\xi_{n,1}^\top)$. Let $|x|$ denote the Euclidean norm of a vector x. We are interested in the asymptotic behavior of $|\tilde{\xi}_n+\mu_n|^2$ with μ_n an r_n-dimensional vector and $\tilde{\xi}_n$ the r_n-dimensional random vector defined by

$$\tilde{\xi}_n = n^{-1/2}\sum_{j=1}^{n}\xi_{n,j}.$$

More precisely, we are looking for conditions that imply the asymptotic normality

$$\frac{|\tilde{\xi}_n+\mu_n|^2-|\mu_n|^2-\mathrm{tr}(V_n)}{\sqrt{2\,\mathrm{tr}(V_n^2)}} \rightsquigarrow \mathcal{N}(0,1). \tag{1}$$

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Of special interest is the case when μ_n is the zero vector and V_n is idempotent with rank q_n tending to infinity. Then (1) simplifies to

$$\frac{|\tilde{\xi}_n|^2 - q_n}{\sqrt{2q_n}} \rightsquigarrow \mathcal{N}(0, 1). \tag{2}$$

In particular, if μ_n is the zero vector and V_n equals I_{r_n}, the $r_n \times r_n$ identity matrix, then (2) becomes

$$\frac{|\tilde{\xi}_n|^2 - r_n}{\sqrt{2r_n}} \rightsquigarrow \mathcal{N}(0, 1). \tag{3}$$

Such results are needed to obtain the asymptotic behavior of the likelihood ratio statistic in exponential families of increasing dimensions and to study the behavior of Owen's empirical likelihood when the data dimension is allowed to increase with the sample size. The former was done by Portnoy [15], who proved (3) under the assumption that the sixth moments of the coordinates of $\xi_{n,1}$ are uniformly bounded. The latter was studied in [9] by Hjort, McKeague and Van Keilegom, who relied on Portnoy's result and in [3] by Chen, Peng and Qin, who relied on results and structural assumptions of [1]. We are interested in verifying (1) under weaker moment assumptions than used by these authors.

The results of this paper play key roles in developing the theory in [14]. Theorem 2 is used there to improve on results in [9], to extend these further to allow for infinitely many constraints that depend on nuisance parameters, and to obtain results similar to those in [3] without their structural assumptions. The results in Section 4 are also used in [14] to address the asymptotic distribution of the empirical likelihood with an increasing number of constraints under contiguous alternatives; see their Remark 2.2 and Remark 7.3. While these applications were the main motivation for the present paper, our results go well beyond this. Several applications to testing problems will be discussed here. The first one focuses on a chi-square test with an increasing number of cells. The second one treats testing for the equality of the marginal distributions of a bivariate random vector. The third one shows that Schott's [19] test for diagonality of the dispersion matrix of a normal random vector remains valid if the dimension of the random vector increases at a faster rate than the sample size, see Theorem 4 in Section 8.

The left-hand side of (1) can be written as the sum $T_{n,1} + T_{n,2} + T_{n,3}$ of the terms

$$T_{n,1} = \frac{1}{n}\sum_{j=1}^{n} \frac{|\xi_{n,j}|^2 - \mathrm{tr}(V_n)}{\sqrt{2\mathrm{tr}(V_n^2)}}, \quad T_{n,2} = \frac{2\tilde{\xi}_n^\top \mu_n}{\sqrt{2\mathrm{tr}(V_n^2)}} \quad \text{and} \quad T_{n,3} = \frac{2}{n}\sum_{1\le i<j\le n} \frac{\xi_{n,i}^\top \xi_{n,j}}{\sqrt{2\mathrm{tr}(V_n^2)}}.$$

The leading term in this expansion is $T_{n,3}$. Indeed, we will give conditions that let us show that $T_{n,1}$ and $T_{n,2}$ converge to zero in probability and let us use a martingale central limit theorem to establish the asymptotic normality of $T_{n,3}$. In contrast to Portnoy [15] who used a martingale central limit theorem to deal with the sum $T_{n,1}+T_{n,3}$, we do so only for the term $T_{n,3}$ and use a different argument to control $T_{n,1}$. Our approach is in the spirit of Guttorp and Lockhart [6], who treated quadratic forms with a random vector of independent components and not necessarily an average. They separated on-diagonal elements and off-diagonal elements into statistics similar to our $T_{n,1}$ and $T_{n,3}$, and treated these two statistics separately relying on work of Rotar [17,16,18]. The use of martingale arguments to deal with quadratic forms with weight matrices with vanishing diagonal is common; see, e.g., the recent work [5].

We achieve our goal by proving two central limit theorems. Both of our theorems require the conditions

$$\mu_n^\top V_n \mu_n = o\{\mathrm{tr}(V_n^2)\} \tag{C1}$$

and

$$\mathrm{tr}(V_n^4) = o\{\mathrm{tr}^2(V_n^2)\}. \tag{C2}$$

Condition (C1) implies that $T_{n,2}$ converges to zero in probability in view of the identity $\mathrm{E}(T_{n,2}^2) = 2\mu_n^\top V_n \mu_n/\mathrm{tr}(V_n^2)$. Condition (C2) is used in the application of the martingale central limit theorem to $T_{n,3}$. This condition already appeared in [4], where one can also find examples of dispersion matrices satisfying (C2). These authors combine (C2) with $\mathrm{tr}(V_n^2) \to \infty$ and the structural assumptions on their random vectors from [1] to derive asymptotic normality of some statistics arising in their tests for diagonality of high-dimensional dispersion matrices.

Of course, (C1) is always met if μ_n is the zero vector. Condition (C2) is met if V_n is idempotent with rank q_n tending to infinity. In particular, it is met if $V_n = I_{r_n}$. In this case, (C1) is implied by $|\mu_n|^2 = o(r_n)$. Condition (C2) is also met if V_n satisfies

$$0 < \lambda = \inf_n \inf_{|u|=1} u^\top V_n u \le \sup_n \sup_{|u|=1} u^\top V_n u = \Lambda < \infty. \tag{4}$$

Indeed, in this case we have $\mathrm{tr}(V_n^4) \le \Lambda^4 r_n$ and $\mathrm{tr}(V_n^2) \ge \lambda^2 r_n$. Under (4), (C1) is implied by $|\mu_n|^2 = o(r_n)$. The sufficiency of (4) for (C2) was already observed in [4]. Finally, in view of the inequality $\mathrm{tr}(V_n^4) \le \rho_n^2 \mathrm{tr}(V_n^2)$ with ρ_n the largest eigenvalue of V_n, a sufficient condition for (C2) is $\rho_n^2 = o\{\mathrm{tr}(V_n^2)\}$. In particular, (C2) holds if ρ_n is bounded and $\mathrm{tr}(V_n^2)$ tends to infinity. These sufficient conditions were used in an earlier version of the present paper, referred to as Peng and Schick (2012) in [14].

The first theorem uses the following growth conditions.

$$\mathrm{var}(|\xi_{n,1}|^2) = o\{n\,\mathrm{tr}(V_n^2)\}, \tag{5}$$