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Strict positive definiteness of multivariate covariance functions on compact two-point homogeneous spaces

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ABSTRACT

The authors provide a characterization of the continuous and isotropic multivariate covariance functions associated to a Gaussian random field with index set varying over a compact two-point homogeneous space. Sufficient conditions for the strict positive definiteness based on this characterization are presented. Under the assumption that the space is not a sphere, a necessary and sufficient condition is given for the continuous and isotropic multivariate covariance function to be strictly positive definite. Under the same assumption, an alternative necessary and sufficient condition is also provided for the strict positive definiteness of a continuous and isotropic bivariate covariance function based on the main diagonal entries in the matrix representation for the covariance function.

1. Introduction

Data observed at locations on a manifold are fairly common. For instance, observations from satellites and output from climate models fit into this framework with the 3-dimensional sphere representing the surface of the Earth. When modeling natural phenomena on the sphere, the existing theory and the known recommendations for fitting spatial models on Euclidean spaces need to be modified or adapted to the spherical setting. In particular, the Euclidean distance between two points in Euclidean space must be replaced by the great circle distance on the sphere, while radial symmetry is replaced with isotropy. These changes are partially justified by the fact that covariance functions defined via an unsuitable distance may lead to models that are physically unrealistic.

In this context, isotropic multivariate covariance functions associated to Gaussian random fields whose index varies over the sphere are crucial and play a central role in the understanding and resolution of many issues in spatial statistics. We mention [8] for a discussion on multivariate covariance functions and a text by Marinucci and Peccati [19] for a modern treatment of the theory of random fields on spheres. Gneiting [10] discusses univariate covariance functions on spheres

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and lists a series of interesting problems related to them. As for continuous isotropic multivariate covariance functions on spheres, their characterization is usually attributed either to E.J. Hannan [11] or to A.M. Yaglom [26]. Some particular classes of isotropic multivariate covariance functions on spheres fitting the Hannan–Yaglom description were studied in [14]. This was followed by [6], with the construction of isotropic variogram matrix functions on spheres, and by [15].

In this paper, we are concerned with the strict and non-strict positive definiteness of a continuous and isotropic multivariate covariance function associated to a Gaussian random field with index set varying over a compact two-point homogeneous space $\mathbb{M}^d$. The upper index d will refer to the dimension of the space, in accordance with the well-known categories established by Wang [24]: the unit spheres S^d, $d = 1, 2, \ldots$, the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$, and the Cayley projective plane $\mathbb{P}^d(Cay)$, $d = 16$. Given that the spheres are compact two-point homogeneous spaces, the results described here are either directly relevant to spatial statistics or pertain to natural versions of results that are so.

Some of the compact two-point homogeneous spaces listed above are relatively complicated and a better understanding of each of them requires the theory of Lie groups. As far as this paper is concerned, we do not need to know what these spaces are, except that there is a major difference between the spheres and those that are not spheres. In a sphere each point possesses just one antipodal point, while in the other compact two-point homogeneous spaces each point possesses a whole manifold of antipodal points. This fact on its own makes the analysis of strict positive definiteness on spheres more complicated than that on the other compact two-point homogeneous spaces. This will become clear in some proofs in the text.

A compact two-point homogeneous space is both a metric space and a quotient group of the form $\mathcal{G}/K$ in which $\mathcal{G}$ is a group of motions and K is the stationary subgroup. If F is a continuous and isotropic multivariate covariance function associated to a Gaussian random field with index set varying over $\mathbb{M}^d$, an *isotropy* of F is characterized by the fact that

$$\forall_{A\in\mathcal{G}} \quad \forall_{x,y\in\mathbb{M}^d} \quad F(Ax, Ay) = F(x, y).$$

In particular, it extends the notion of isotropy typically used in the spherical setting.

Common references where compact two-point homogeneous spaces are used as index sets for covariance functions include [17,18,16]. In the first one, the main theme is the relationship between the asymptotic behavior of the spectral measure of the random field near infinity and the asymptotic behavior of the incremental variance near zero. The third one studies local properties of sample functions of certain Gaussian random fields on $\mathbb{M}^d$ while the second is a textbook on spectral theory of invariant random fields, with a whole chapter dealing with Gaussian random fields.

Applications of multivariate covariance functions in spatial statistics are mainly focused on interpolation and smoothing. The best linear unbiased predictor depends on the inverse of matrices defined by the covariance function at some scattered data points, the invertibility of which is guaranteed by the strict positive definiteness of the covariance function. In a general framework as the one considered here, strict positive definiteness guarantees that certain interpolation procedures related to scattered data at arbitrary points on the index set be uniquely solved.

The paper proceeds as follows. In Section 2, we introduce a bit of notation and recall a few concepts related to the material to be developed in the paper. In Section 3, we detail an extension of the Hannan–Yaglom result to all the compact two-point homogeneous spaces. Section 4 addresses the main objective of the paper, namely a characterization for the strict positive definiteness of a continuous and isotropic multivariate covariance function associated to a Gaussian field with index set varying over $\mathbb{M}^d$ under the assumption that $\mathbb{M}^d$ is not a sphere. The characterization itself is an interesting result but, in a certain sense, it is still an abstract criterion, since it does not involve the entry functions in the matrix representation of the covariance function. For that reason, we change direction in Section 5 and obtain direct sufficient conditions for strict positive definiteness, without any restriction on the space $\mathbb{M}^d$. Finally, for a continuous and isotropic bivariate covariance function on $\mathbb{M}^d$, the results in Section 5 imply a characterization for strict positive definiteness based on the main diagonal entries in the matrix representation of the function; details are provided in Section 6.

2. Background and notation

From now on, a multivariate covariance function associated to a Gaussian random field with index set varying over $\mathbb{M}^d$ will be just called a *multivariate covariance function on* $\mathbb{M}^d$. For a fixed positive integer ℓ, denote by $M_\ell(\mathbb{R})$ the set of all $\ell \times \ell$ matrices with real entries. An *ℓ-variate covariance function* on $\mathbb{M}^d$ is then a matrix function $F : \mathbb{M}^d \times \mathbb{M}^d \to M_\ell(\mathbb{R})$ with the following additional features:

$$\forall_{\mu,\nu\in\{1,\ldots,\ell\}} \quad \forall_{x,y\in\mathbb{M}^d} \quad f_{\mu\nu}(x, y) = f_{\mu\nu}(y, x) = f_{\nu\mu}(y, x),$$

and if

$$\forall_{x,y\in\mathbb{M}^d} \quad F(x, y) = [f_{\mu\nu}(x, y)]_{\mu,\nu=1}^{\ell},$$

then for each positive integer n and for each choice of n distinct points $x_1, \ldots, x_n$ on $\mathbb{M}^d$, the $\ell n \times \ell n$ block matrix $[f_{\mu\nu}(x_i, x_j)]$ is nonnegative definite in the sense that

$$\sum_{i,j=1}^{n} B_i^t F(x_i, x_j) B_j = \sum_{i,j=1}^{n} \sum_{\mu,\nu=1}^{\ell} B_i^{\mu} B_j^{\nu} f_{\mu\nu}(x_i, x_j) \geq 0,$$