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Multivariate Archimax copulas

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ABSTRACT

A multivariate extension of the bivariate class of Archimax copulas was recently proposed by Mesiar and Jágr (2013), who asked under which conditions it holds. This paper answers their question and provides a stochastic representation of multivariate Archimax copulas. A few basic properties of these copulas are explored, including their minimum and maximum domains of attraction. Several non-trivial examples of multivariate Archimax copulas are also provided.

1. Introduction

A d-variate copula is the joint cumulative distribution function of a vector $(U_1, \ldots, U_d)$ of random variables each having a uniform distribution on the interval $(0, 1)$. Following Capéraà et al. [5], a bivariate copula is said to be Archimax if it can be written, for all $u_1, u_2 \in (0, 1)$, in the form

$$C_{\psi,A}(u_1, u_2) = \psi\left[\{\psi^{-1}(u_1) + \psi^{-1}(u_2)\}A\left\{\frac{\psi^{-1}(u_1)}{\psi^{-1}(u_1) + \psi^{-1}(u_2)}\right\}\right], \tag{1}$$

using maps $A : [0, 1] \to [1/2, 1]$ and $\psi : [0, \infty) \to [0, 1]$ such that

(i) A is convex and, for all $t \in [0, 1]$, $\max(t, 1 - t) \leq A(t) \leq 1$;
(ii) ψ is convex, decreasing and such that $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$, with the convention that $\psi^{-1}(0) = \inf\{x \geq 0 : \psi(x) = 0\}$.

The term Archimax was chosen by Capéraà et al. [5] to reflect the fact that if $A \equiv 1$, $C_{\psi,A}$ reduces to an Archimedean copula, viz.

$$C_\psi(u_1, u_2) = \psi\{\psi^{-1}(u_1) + \psi^{-1}(u_2)\}$$

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while if $\psi(t) = e^{-t}$ for all $t \in [0, \infty)$, $C_{\psi,A}$ is an extreme-value copula, viz.

$$C_A(u_1, u_2) = \exp\left[\ln(u_1u_2)A\left\{\frac{\ln(u_1)}{\ln(u_1u_2)}\right\}\right].$$

In [5], Archimax copulas are presented as a tool for constructing bivariate distribution functions in the maximum domain of attraction of an extreme-value copula $C_{A^\star}$ where, for all $t \in (0, 1)$,

$$A^\star(t) = \{t^{1/\alpha} + (1-t)^{1/\alpha}\}^\alpha A^\alpha\left\{\frac{t^{1/\alpha}}{t^{1/\alpha} + (1-t)^{1/\alpha}}\right\} \tag{2}$$

when the map $t \mapsto \psi^{-1}(1 - 1/t)$ is regularly varying at infinity of degree $-1/\alpha$ with $\alpha \in (0, 1]$; see, e.g., p. 13 in [25] for a definition of regular variation. Bivariate Archimax copulas have been further studied and found to be useful in various contexts since their introduction; see, e.g., [1] for applications in hydrology and www.math.sk/wiki/bacigal for a library of R programs.

Recently, Bacigál and Mesiar [2] and Mesiar and Jágr [21] proposed an extension of the family (1) to arbitrary dimension $d \geq 3$. Their generalization involves the notion of stable tail dependence function originally due to Huang [12]. A function $\ell : [0, \infty)^d \to [0, \infty)$ is called a d-variate stable tail dependence function if there exists a d-variate extreme-value copula D such that, for all $x_1, \ldots, x_d \in [0, \infty)$,

$$\ell(x_1, \ldots, x_d) = -\ln\{D(e^{-x_1}, \ldots, e^{-x_d})\}.$$

Let $\psi : [0, \infty) \to [0, 1]$ be the generator of a d-variate Archimedean copula C_ψ defined, for all $u_1, \ldots, u_d \in (0, 1)$, by

$$C_\psi(u_1, \ldots, u_d) = \psi\{\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\}.$$

As shown by McNeil and Nešlehová [19], this occurs if and only if the map $\psi : [0, \infty) \to [0, 1]$ satisfies $\psi(0) = 1$, $\lim_{x\to\infty} \psi(x) = 0$ and is d-monotone. The latter property means that ψ has $d - 2$ derivatives on $(0, \infty)$ and, for all $j \in \{0, \ldots, d - 2\}$, $(-1)^j\psi^{(j)} \geq 0$ with $(-1)^{d-2}\psi^{(d-2)}$ being non-increasing and convex on $(0, \infty)$.

Mesiar and Jágr [21] suggest that a suitable d-variate extension of the notion of Archimax copula would be obtained by setting, for all $u_1, \ldots, u_d \in (0, 1)$,

$$C_{\psi,\ell}(u_1, \ldots, u_d) = \psi \circ \ell\{\psi^{-1}(u_1), \ldots, \psi^{-1}(u_d)\}. \tag{3}$$

This is indeed reasonable, as when $d = 2$, one recovers Eq. (1) by setting $A(t) = \ell(t, 1-t)$ for all $t \in [0, 1]$. While expression (3) appears as formula (18) in [21], these authors merely conjecture that $C_{\psi,\ell}$ is a copula for any choice of ψ and ℓ. This is their Open Problem 4.1.

The purpose of this paper is to solve this problem by showing that $C_{\psi,\ell}$ as defined above is indeed a copula for any combination of d-variate Archimedean generator ψ and d-variate stable tail dependence function ℓ. This result is established in Section 2 by combining a composition theorem of Morillas [22], a characterization of d-variate Archimedean generators popularized by McNeil and Nešlehová [19], and a recent characterization of stable tail dependence functions due to Ressel [26]. Two different stochastic representations of multivariate Archimax copulas are then provided in Section 3 which shed light on their properties and facilitate simulation; it is also emphasized there that for some d-variate stable tail dependence functions ℓ, the condition on ψ is not necessary for (3) to be a copula. Algorithms for generating observations from this new class of copulas are presented in Section 4. These results are illustrated in Section 5 using new and existing examples of Archimax copulas. The maximum and minimum attractors of multivariate Archimax copula families are then derived in Section 6. Finally, a few remaining challenges are outlined in Section 7, where partial results are offered on the level of dependence that can be achieved by multivariate Archimax copulas.

2. Eq. (3) defines bona fide copulas

The purpose of this section is to show the following result, which solves Open Problem 4.1 in [21].

Theorem 2.1. *Let ℓ be a d-variate stable tail dependence function and ψ be the generator of a d-variate Archimedean copula. There exists a vector $(X_1, \ldots, X_d)$ of strictly positive random variables such that, for all $x_1, \ldots, x_d \in [0, \infty)$,*

$$\Pr(X_1 > x_1, \ldots, X_d > x_d) = \psi \circ \ell(x_1, \ldots, x_d).$$

In particular, $\Pr(X_j > x_j) = \psi(x_j)$ for all $x_j \in [0, \infty)$ and $j \in \{1, \ldots, d\}$.

The proof of this proposition relies on the following recent characterization of stable tail dependence functions due to Ressel [26]. In what follows, $\mathbf{e}_j$ denotes a d-dimensional vector whose components are all 0 except the jth, which is equal to 1. Furthermore, $\mathbf{1}_A$ denotes the indicator of the event A.

Theorem 2.2 (*Ressel, 2013*)**.** *A function $\ell : [0, \infty)^d \to [0, \infty)$ is a d-dimensional stable tail dependence function if and only if*
(a) *ℓ is homogeneous of degree 1, i.e., for all $k \in (0, \infty)$ and $x_1, \ldots, x_d \in [0, \infty)$, $\ell(kx_1, \ldots, kx_d) = k\,\ell(x_1, \ldots, x_d)$;*
(b) *$\ell(\mathbf{e}_1) = \cdots = \ell(\mathbf{e}_d) = 1$;*
(c) *ℓ is fully d-max decreasing, i.e., for all $x_1, \ldots, x_d, h_1, \ldots, h_d \in [0, \infty)$ and any $J \subseteq \{1, \ldots, d\}$ of arbitrary size $|J| = k$,*

$$\sum_{\iota_1,\ldots,\iota_k\in\{0,1\}} (-1)^{\iota_1+\cdots+\iota_k}\ell(x_1 + \iota_1h_1\mathbf{1}_{1\in J}, \ldots, x_d + \iota_dh_d\mathbf{1}_{d\in J}) \leq 0.$$