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Bayesian optimality for Beran's class of tests of uniformity around the circle

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ABSTRACT

We show that the locally most powerful tests of uniformity on the circle given by Beran have optimality properties not just against the parametric alternatives described by Beran but also against many non-parametric alternatives. We compare these local results to asymptotic versions of this optimality.

1. Introduction

Suppose we have a sample of observations located on the circumference of a circle and want to test if these are randomly distributed with no preferred directions. For example, in a study of habitat perceptions in tropical butterflies, conducted by Cardoso et al. (2017), butterflies were transported from their home habitat to an open field a certain distance away and then released. The butterflies' flying directions were recorded as points on a unit circle. A test of randomness in the directions can then be used to decide if butterflies can perceive direction.

Testing for randomness around a circle (also called testing for uniformity) is therefore an important problem in the field of directional data. Many tests have been developed for this purpose. Among them are a class of tests, introduced by Beran (1968, 1969), which are locally most powerful invariant for testing uniformity around the circle against a certain parametric alternative. Many classical tests belong to this class. Examples include Rayleigh's test, Watson's U^2 test (Watson, 1961), Ajne's test (Ajne, 1968), Rothman–Rao's test (Rothman, 1972; Rao, 1976) and more recently, Pycke's test (Pycke, 2010).

In Contreras et al. (2017), Bayesian priors are used to study the average power of goodness-of-fit tests over a band of alternative distributions around the null distribution. The band has a sample-size dependent width chosen so that distances between alternative distributions in the band and the null are large enough to be detectable but small enough not to be obvious; that is, the priors are supported on contiguous neighbourhoods of the null. The Neyman–Pearson Lemma is then used to find an asymptotically (Bayes) optimal test.

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In this paper we define Bayes average power of tests of uniformity on the circle using priors on the family of alternative densities. We re-interpret Beran's result to show that his tests are, for fixed sample size n, locally optimal for Bayes power for a certain parametric prior distribution. Then we describe general non-parametric priors on the alternative derived by treating the log density as a stochastic process. We show that Beran's class provides tests which are locally most powerful Bayes against the non-parametric alternative represented by our priors. Finally we interpret the results of Contreras et al. to show that tests in Beran's class are asymptotically optimal in a non-local framework.

Our proposals are focused on the frequency theory properties of tests but they should be compared to truly Bayesian methods in goodness-of-fit. Nonparametric Bayesian goodness-of-fit testing considers the null hypothesis that a distribution belongs to a finite dimensional parametric model. This parametric model is embedded in a larger family whose number of parameters can grow as the sample size grows or is infinite so that the alternative is infinite-dimensional (nonparametric).

A common Bayesian testing procedure is to assign a prior on the null and alternative and compute the Bayes factor. Verdinelli and Wasserman (1998) developed a nonparametric Bayes factor for goodness-of-fit and examined the consistency of the procedure; their prior is closely connected to ours. See Delampady and Berger (1990), Berger and Pericchi (1996) and Kass and Raftery (1995) for more goodness of fit procedures based on Bayes factors. Relatively speaking, the literature on nonparametric Bayesian goodness of fit is still small; see Tokdar et al. (2010) for a review paper on this subject.

In Section 2 we describe Beran's results. In Section 3 we define Bayes optimal tests and locally most powerful Bayes tests and re-interpret Beran's result as showing that his tests are locally most powerful Bayes for a wide variety of priors. In Section 4 we discuss briefly the non-local asymptotic version of these results described in detail in Contreras et al. (2017). In Section 5 specific tests fitting in this framework are described. Section 6 has discussion and final remarks. Proofs are in Section 7.

2. Beran's tests

Consider a sample $P_1, \ldots, P_n$ of points on the unit circle. Pick an arbitrary ray and let $0 \le 2\pi U_i < 2\pi$ be the angle between the ray and the point P_i. The null hypothesis H_o that the sample is uniformly distributed around the circle is equivalent to the hypothesis that the U_i are a sample from the uniform density, $f_0(u) = 1, 0 \le u \le 1$. A test is a function, ψ, mapping $[0, 1]^n$ to $[0, 1]$. Its power against a density f is

$$\mathrm{Pow}(\psi, f) = \mathrm{E}_f \{\psi(U_1, \ldots, U_n)\} = \int \psi(u_1, \ldots, u_n) \prod_1^n f(u_i) du_1 \cdots du_n.$$

Beran considers the alternative that $U_1, \ldots, U_n$ are iid with density

$$f_\epsilon(u|\theta) = 1 - \epsilon + \epsilon g(u - \theta) = 1 + \epsilon \{g(u - \theta) - 1\}$$

for some $\epsilon > 0$, some fixed density $g \in L_2[0, 1]$ and some unknown parameter $\theta \in [0, 1)$. Here and throughout the paper addition and subtraction of points in [0,1], representing points on the circle, are interpreted modulo 1.

Since the choice of the axis from which angles are measured was arbitrary it is natural to require that tests be invariant to the choice of this axis. Formally we restrict the class of test functions ψ to those such that

$$\psi(U_1 + a, \ldots, U_n + a) = \psi(U_1, \ldots, U_n).$$

Let Ψ be the set of all such test functions. The power of any invariant test ψ against $f_\epsilon(u|\theta)$ is easily seen not to depend on θ. In what follows we therefore write $\mathrm{Pow}(\psi, f_\epsilon)$ rather than $\mathrm{Pow}\{\psi, f_\epsilon(\cdot|\theta)\}$.

Define

$$h(u) = \int_0^1 \{g(u - \theta) - 1\} \{g(-\theta) - 1\} d\theta. \tag{1}$$

Beran's test statistic is

$$B_n = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n h(U_i - U_j).$$

Beran proves

Theorem 1. *For $0 \le \epsilon \le 1$ the power of any invariant test against the alternative f_ϵ may be written in the form*

$$\mathrm{Pow}(\psi, f_\epsilon) = \mathrm{Pow}(\psi, f_0) + \epsilon^2 C(\psi) + \epsilon^3 R(\epsilon, \psi)$$

where

1. *The remainder term, $R(\epsilon, \psi)$ is bounded over all $\epsilon \in [0, 1]$ and all invariant tests ψ. In fact*

$$|R(\epsilon, \psi)| \le \sum_{r=3}^n \epsilon^{r-3} \binom{n}{r} \|g - 1\|_1^r \le \sum_{r=3}^n \binom{n}{r} \|g - 1\|_1^r$$

where $\| \cdot \|_1$ denotes L_1 norm.