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Peakedness and convex ordering for elliptically contoured random fields

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ABSTRACT

For the peakedness comparison between two Gaussian random fields about their mean functions, a necessary and sufficient condition is derived in this paper in terms of their covariance functions. Interestingly, such a condition is also necessary and sufficient for the convex ordering between the two Gaussian random fields having identical mean functions. The relation to the equivalence of two Gaussian random fields is illustrated through some parametric examples. Necessary and/or sufficient conditions are given for the peakedness comparison and convex ordering between two elliptically contoured random fields. These conditions are applied to examine how certain parameters affect the peakedness of some Gaussian or elliptically contoured random fields.

1. Introduction

Suppose that $\{Z_1(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\}$ are two real random fields whose finite-dimensional distributions are symmetric about $\mu_1(x)$ and $\mu_2(x)$, respectively, where $\mathbb{D}$ is a temporal, spatial, or spatio-temporal index domain. We say that $\{Z_1(x), x \in \mathbb{D}\}$ is more peaked about $\mu_1(x)$ than $\{Z_2(x), x \in \mathbb{D}\}$ about $\mu_2(x)$, and denote it by $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$, if

$$\begin{aligned} &P((Z_1(x_1) - \mu_1(x_1), \ldots, Z_1(x_n) - \mu_1(x_n))' \in A_n) \\ &\geq P((Z_2(x_1) - \mu_2(x_1), \ldots, Z_2(x_n) - \mu_2(x_n))' \in A_n), \end{aligned} \tag{1.1}$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}\,(k = 1, \ldots, n)$, and any $A_n \in \mathscr{A}_n$, where $\mathbb{N}$ is the set of positive integers, and $\mathscr{A}_n$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^n$. In particular, $A_n = [-z_1, z_1] \times \cdots \times [-z_n, z_n] \in \mathscr{A}_n$, and inequality (1.1) reads

$$\begin{aligned} &P(|Z_1(x_1) - \mu_1(x_1)| \leq z_1, \ldots, |Z_1(x_n) - \mu_1(x_n)| \leq z_n) \\ &\geq P(|Z_2(x_1) - \mu_2(x_1)| \leq z_1, \ldots, |Z_2(x_n) - \mu_2(x_n)| \leq z_n), \\ &\qquad z_1, \ldots, z_n \geq 0. \end{aligned} \tag{1.2}$$

More specifically, for $n = 1$, it reads

$$P(|Z_1(x) - \mu_1(x)| \geq z) \leq P(|Z_2(x) - \mu_2(x)| \geq z), \qquad z \geq 0, \tag{1.3}$$

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and such a peakedness order, introduced by Birnbaum (1948), is a variability notation that applies to random variables with symmetric distribution functions, and it stochastically compares random variables according to their distance from their center of symmetry (Olkin and Tong, 1988). Alternatively, (1.3) means that the random variable $|Z_1(x) - \mu_1(x)|$ is smaller than the random variable $|Z_2(x) - \mu_2(x)|$ in the usual stochastic order; in symbol, $|Z_1(x) - \mu_1(x)| \preceq_{st} |Z_2(x) - \mu_2(x)|$. For properties and applications of the peakedness and stochastic orders, we refer the reader to Dharmadhikari and Joag-Dev (1988), Marshall et al. (2011), Müller and Stoyan (2002), Olkin and Tong (1988), Shaked and Shanthikumar (2007) and Tong (1990), among others.

A real function $g(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is convex, if

$$g(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \le \lambda g(\mathbf{x}_1) + (1-\lambda) g(\mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n,$$

holds for any $\lambda \in [0, 1]$. We say that $\{Z_1(x), x \in \mathbb{D}\}$ is smaller than $\{Z_2(x), x \in \mathbb{D}\}$ in the convex order, denoted by $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$, if the inequality

$$\mathrm{E}g(Z_1(x_1), \ldots, Z_1(x_n)) \le \mathrm{E}g(Z_2(x_1), \ldots, Z_2(x_n)) \tag{1.4}$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ $(k = 1, 2, \ldots, n)$, and any convex function $g(\mathbf{z})$ such that the expected values on both sides of (1.4) exist. The main goal of this paper is to compare elliptically contoured random fields through the peakedness and convex ordering.

An elliptically contoured (or spherically invariant) random field is a scale mixture of Gaussian random fields, and its finite-dimensional distributions are symmetric about the center (Huang and Cambanis, 1979; Ma, 2009, 2011; Yao, 2003). More precisely, $\{Z(x), x \in \mathbb{D}\}$ is called an elliptically contoured random field, if it can be expressed as

$$Z(x) = UZ_0(x) + \mu(x), \quad x \in \mathbb{D}, \tag{1.5}$$

where $\{Z_0(x), x \in \mathbb{D}\}$ is a Gaussian random field with mean 0, U is a nonnegative random variable and is independent of $\{Z_0(x), x \in \mathbb{D}\}$, and $\mu(x)$ is a (non-random) function. Examples of elliptically contoured random fields include, but are not limited to, Gaussian, Student's t (Ma, 2013a; Røislien and Omre, 2006), logistic, hyperbolic (Du et al., 2012), Mittag-Leffler, Linnik, stable, and Laplace ones. An elliptically contoured random field may or may not have first-order moments, but its finite-dimensional distributions are symmetric about its center. Among all second-order random fields, the class of second-order elliptically contoured random fields is one of the largest, if not the largest, classes that allow for any given correlation structure (Ma, 2013a).

In Section 2, we provide a necessary and sufficient condition for the peakedness comparison about their mean functions of two Gaussian random fields through their covariance structures, and show that such a condition is also necessary and sufficient to make the convex ordering between the two Gaussian random fields with the same mean functions. Sufficient and/or necessary conditions are obtained in Section 3 to compare the peakedness and convex order of two elliptically contoured random fields. These conditions are applied to investigate how certain parameters affect the peakedness of some Gaussian or elliptically contoured random fields. Proofs of theorems are given in Section 4.

2. Peakedness comparison for Gaussian random fields

A Gaussian random field is characterized by its mean and covariance functions. In this section we compare the peakedness of two Gaussian random fields about their mean functions through their covariance structures, for which a necessary and sufficient condition is given in Theorem 1. More interestingly, such a condition is equivalent to that makes the convex ordering between the two Gaussian random fields with the same mean functions, as is shown in Theorem 2.

Theorem 1. *Suppose that $\{Z_k(x), x \in \mathbb{D}\}$ is a Gaussian random field with mean function $\mu_k(x)$ and covariance function $C_k(x_1, x_2)$ $(k = 1, 2)$. Then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ if and only if $C_2(x_1, x_2) - C_1(x_1, x_2)$ is the covariance function of another Gaussian random field.*

Example 1. Consider two Gaussian random fields $\{Z_k(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$ with Matérn or von Kármán–Whittle covariance functions (Anderes, 2011; Du et al., 2009; Wang and Loh, 2011)

$$C_k(\mathbf{x}_1, \mathbf{x}_2) = \beta_k (\alpha_k \|\mathbf{x}_1 - \mathbf{x}_2\|)^\nu K_\nu(\alpha_k \|\mathbf{x}_1 - \mathbf{x}_2\|), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,$$

where ν, α_k, β_k are positive constants $(k = 1, 2)$, $K_\nu(x)$ is the modified Bessel function of the second type of order ν, and $\|\mathbf{x}_1 - \mathbf{x}_2\|$ is the usual Euclidean distance between $\mathbf{x}_1$ and $\mathbf{x}_2$. In order that $\{Z_1(\mathbf{x}) - \mu_1(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\} \overset{p}{\succeq} \{Z_2(\mathbf{x}) - \mu_2(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$, it is necessary and sufficient that $C_2(\mathbf{x}_1, \mathbf{x}_2) - C_1(\mathbf{x}_1, \mathbf{x}_2)$ is positive definite by Theorem 1, or, equivalently, its Fourier transform is nonnegative in $\mathbb{R}^d$ by Bochner's theorem. The latter is the difference between the Fourier transform of $C_2(\mathbf{x}_1, \mathbf{x}_2)$ and that