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D-optimal designs for complex Ornstein–Uhlenbeck processes

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ABSTRACT

Complex Ornstein–Uhlenbeck (OU) processes have various applications in statistical modelling. They play role e.g. in the description of the motion of a charged test particle in a constant magnetic field or in the study of rotating waves in time-dependent reaction diffusion systems, whereas Kolmogorov used such a process to model the so-called Chandler wobble, small deviation in the Earth's axis of rotation. In these applications parameter estimation and model fitting is based on discrete observations of the underlying stochastic process, however, the accuracy of the estimation strongly depend on the observation points.

This paper studies the properties of D-optimal designs for estimating the parameters of a complex OU process with a trend. In special situations we show that in contrast with the case of the classical real OU process, a D-optimal design exists not only for the trend parameter, but also for joint estimation of the covariance parameters, moreover, these optimal designs are equidistant.

1. Introduction

Random processes have various applications in statistical modelling in different areas of science such as physics, chemistry, biology or finance, where one usually cannot observe continuous trajectories. In these situations parameter estimation and model fitting is based on discrete observations of the underlying stochastic process, however, the accuracy of the results strongly depend on the observation points. The theory of optimal experimental designs, dating back to the late 50s of the twentieth century (see e.g. Hoel, 1958; Kiefer, 1959), deals with finding design sets $\boldsymbol{\xi} = \{t_1, t_2, \dots, t_n\}$ of distinct time points (or locations in space) where the process under study is observed, which are optimal according to some previously specified criterion (Müller, 2007). In parameter estimation problems the most popular criteria are based on the Fisher information matrix (FIM) of the observations. D-, E- and T-optimal designs maximize the determinant, the smallest eigenvalue and the trace of the FIM, respectively, an A-optimal design minimizes the trace of the inverse of the FIM (for an overview see Pukelsheim, 1993), whereas K-optimality refers to the minimization of the condition number of the FIM (see e.g. Ye and Zhou, 2013; Baran, 2017). In the last decades information based criteria have intensively been studied both in the uncorrelated setup (see e.g. Silvey, 1980) and in the more difficult correlated situation (Dette et al., 2015, 2016).

In the present paper we derive D-optimal designs for parameter estimation of complex (or vector) Ornstein–Uhlenbeck (OU) processes with trend (see e.g. Arató, 1982), defined in detail in Section 2. A complex OU process describes e.g. the motion of a charged test particle in a constant magnetic field (Balescu, 1997), it is used in the description of the rotation of a planar

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polymer (Vakeroudis et al., 2011) or in the study of rotating waves in time-dependent reaction diffusion systems (Beyn and Lorenz, 2008; Otten, 2015), and it also has several applications in financial mathematics (see e.g. Barndorff-Nielsen and Shephard, 2001). Further, Kolmogorov proposed to model the so-called Chandler wobble, small deviation in the Earth's axis of rotation (Lambeck, 1980), by the model

$$Z(t) = Z_1(t) + iZ_2(t) = me^{i2\pi t} + Y(t), \qquad t > 0, \tag{1.1}$$

where $Z_1(t)$ and $Z_2(t)$ are the coordinates of the deviation of the instantaneous pole from the North Pole and $Y(t)$ is a complex OU process (Arató et al., 1962). We remark that most of our results correspond to the special case of a constant trend, however, even this simple situation gives a nice insight into the behavior of D-optimal designs for complex OU processes, highlighting the differences between the real and complex models.

Note that the properties of D-optimal designs for classical one-dimensional OU processes have already investigated by Kiseľák and Stehlík (2008) and later by Zagoraiou and Baldi Antognini (2009), where the authors proved that there is no D-optimal design for estimating the covariance parameter, whereas the D-optimal design for trend estimation is equidistant and larger distances resulting in more information. Later these results were generalized for OU sheets under various sampling schemes (Baran and Stehlík, 2015; Baran et al., 2013, 2015).

The paper is organized as follows. In Section 2 we introduce the model to be studied, Section 3 contains our results on D-optimal designs, whereas in Section 4 some applications are presented. The paper ends with the concluding remarks of Section 4. To maintain the continuity of the explanation, the proofs are given in the Appendix.

2. Complex Ornstein–Uhlenbeck process with a trend

Consider the complex stochastic process $Z(t) = Z_1(t) + iZ_2(t)$, defined as

$$Z(t) = mf(t) + Y(t), \qquad t \geq 0, \tag{2.1}$$

with design points taken from the non-negative half-line $\mathbb{R}_+$, where $m = m_1 + im_2,\ m_1, m_2 \in \mathbb{R}, f(t) = f_1(t) + if_2(t)$ with $f_1(t), f_2(t) : \mathbb{R}_+ \mapsto \mathbb{R}$ and $Y(t) = Y_1(t) + iY_2(t),\ t \geq 0$, is a complex valued stationary OU process. The process $Y(t)$ can be defined by the stochastic differential equation (SDE)

$$\mathrm{d}Y(t) = -\gamma Y(t)\mathrm{d}t + \sigma \mathrm{d}\mathcal{W}(t), \qquad Y(0) = \xi, \tag{2.2}$$

where $\gamma = \lambda - i\omega$ with $\lambda > 0,\ \omega \in \mathbb{R}, \sigma > 0, \mathcal{W}(t) = \mathcal{W}_1(t) + i\mathcal{W}_2(t),\ t \geq 0$, is a standard complex Brownian motion, that is $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ are independent standard Brownian motions, and $\xi = \xi_1 + i\xi_2$, where ξ_1 and ξ_2 are centered normal random variables that are chosen according to stationarity (Arató, 1982).

Instead of the complex process $Y(t)$ defined by (2.2) one can consider the two-dimensional real valued stationary OU process $\big(Y_1(t), Y_2(t)\big)^\top$ defined by the SDE

$$\begin{bmatrix}\mathrm{d}Y_1(t)\\ \mathrm{d}Y_2(t)\end{bmatrix} = A\begin{bmatrix}Y_1(t)\\ Y_2(t)\end{bmatrix}\mathrm{d}t + \sigma\begin{bmatrix}\mathrm{d}\mathcal{W}_1(t)\\ \mathrm{d}\mathcal{W}_2(t)\end{bmatrix}, \quad \text{where} \quad A := \begin{bmatrix}-\lambda & -\omega\\ \omega & -\lambda\end{bmatrix}. \tag{2.3}$$

We remark that in physics (2.3) is called A-Langevin equation, see e.g. Balescu (1997). If $\big(Y_1(t), Y_2(t)\big)^\top$ satisfies (2.3) then $Y_1(t) + iY_2(t)$ is a complex OU process which solves (2.2), and conversely, the real and imaginary parts of a complex OU process form a two-dimensional real OU process satisfying (2.3). Obviously, $\mathsf{E}Y_1(t) = \mathsf{E}Y_2(t) = 0$, whereas the covariance matrix function of the process $\big(Y_1(t), Y_2(t)\big)^\top$ is given by

$$\mathcal{R}(\tau) := \mathsf{E}\begin{bmatrix}Y_1(t+\tau)\\ Y_2(t+\tau)\end{bmatrix}\begin{bmatrix}Y_1(t)\\ Y_2(t)\end{bmatrix}^\top = \frac{\sigma^2}{2\lambda}\mathrm{e}^{A\tau} = \frac{\sigma^2}{2\lambda}\mathrm{e}^{-\lambda\tau}\begin{bmatrix}\cos(\omega\tau) & -\sin(\omega\tau)\\ \sin(\omega\tau) & \cos(\omega\tau)\end{bmatrix}, \quad \tau \geq 0. \tag{2.4}$$

This results in a complex covariance function of the complex OU process $Y(t)$ defined by (2.2) of the form

$$\mathcal{C}(\tau) := \mathsf{E}Y(t+\tau)\overline{Y(t)} = \frac{\sigma^2}{\lambda}\mathrm{e}^{-\lambda\tau}\big(\cos(\omega\tau) + i\sin(\omega\tau)\big), \qquad \tau \geq 0,$$

behaving like a damped oscillation with frequency ω.

In the present study the damping parameter λ, frequency ω and standard deviation σ are assumed to be known. However, a valuable direction for future research will be the investigation of models where these parameters should also be estimated. Note that the estimation of σ can easily be done on the basis of a single realization of the complex process, see e.g. Arató (1982, Chapter 4). Now, without loss of generality, one can set the variances of $Y_1(t)$ and $Y_2(t)$ to be equal to one, that is $\sigma^2/(2\lambda) = 1$, which reduces $\mathcal{R}(\tau)$ to a correlation matrix function. Further results on the maximum-likelihood estimation of the covariance parameters can be found e.g. in Arató et al. (1999).

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