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A general result on the estimation bias of ARMA models

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ABSTRACT

A general result is derived on the finite-sample approximate bias of the Gaussian maximum likelihood estimator of the full vector of parameters in ARMA models when the error term may be nonnormally distributed and exogenous regressors may be included. It is found that there is typically estimation bias for parameters associated with the AR and MA terms and the bias depends only on these parameters themselves and the exogenous regressors. Parameters associated with the exogenous regressors are estimated approximately unbiased. The error variance is estimated with a downward bias, proportional to the number of exogenous regressors and the orders of AR and MA terms in the model. The distributional assumption on the error term does not affect the general bias result. Monte Carlo experiments are provided to illustrate the effectiveness of using analytical bias for the purpose of bias correction.

1. Introduction

More than thirty years ago, Tanaka (1984) discussed a technique for obtaining the Edgeworth type asymptotic expansion of the maximum likelihood estimator (MLE) in autoregressive moving average (ARMA) models. With the expansion in hand, he was able to derive analytically the bias of the MLE in ten ARMA models.[2] Ten years later, Cordeiro and Klein (1994), following a somewhat different approach of Cox and Snell (1968), were able to verify the bias results for three of the ten models in Tanaka (1984).

There has been a long literature documenting the finite-sample bias in estimating ARMA models (and no attempt is made here to list all the references in this literature). Among all the existing works, two observations may be made. Firstly, it appears that no unified approach was used and different authors worked on different specific ARMA models. Typically, they were simple ARMA models of low orders. The most discussed model may be the AR(1) model, see Kendall (1954), White (1961), Shenton and Johnson (1965) and Phillips (1977), among others. Tanaka (1984) may be the first one to provide a table of bias results for ten ARMA models, some of which were never discovered prior to his work. Yet, a general bias result for an ARMA model of any order was not available. Secondly, in deriving the bias results for these specific ARMA models, most authors imposed normality on the data generating processes. In social sciences, due to the nature of data, normality is an exception rather than a norm, and the sample sizes are usually quite limited. Thus, one may wonder whether this stringent distributional assumption could be relaxed in deriving the finite-sample bias.

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[2] Throughout, the word "bias" refers to the approximate bias, not the exact bias. That is, given a sample of size T, for a parameter vector $\boldsymbol{\alpha}$ and its estimator $\hat{\boldsymbol{\alpha}}$, suppose one has a stochastic expansion $\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} = \boldsymbol{a}_{-1/2} + \boldsymbol{a}_{-1} + \boldsymbol{r}_T$, where $\boldsymbol{a}_{-1/2}$ and $\boldsymbol{a}_{-1}$ are $O_P(T^{-1/2})$ and $O_P(T^{-1})$, respectively, and the remainder term $\boldsymbol{r}_T$ is $o_P(T^{-1})$. Assuming that $\boldsymbol{r}_T$ is uniformly integrable, one can define the approximate bias of $\hat{\boldsymbol{\alpha}}_T$, up to $O(T^{-1})$, as $\mathrm{E}(\boldsymbol{a}_{-1/2} + \boldsymbol{a}_{-1})$. It may happen that the exact moment of $\hat{\boldsymbol{\alpha}}$ does not exist and hence its bias is not defined. In this case, one may interpret the approximate bias as the moment of the approximating distribution of $\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}$ that does have moments. More detailed discussions are provided in Sargan (1974).

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Bao (2016) recently revisited this area of research by outlining the numerical algorithm for calculating the finite-sample bias of the Gaussian MLE (or quasi MLE, QMLE for short) in ARMA models without exogenous regressors. By studying specific ARMA models and comparing with Tanaka (1984), Bao (2016) verified most of the bias results in Tanaka (1984) and also found some discrepancies regarding the ARMA(1,1) model.[3] Simulation results suggest that the bias approximation in Bao (2016) was most reliable whereas Tanaka (1984) could give very misleading results for ARMA(1,1). Moreover, based on the ten models studied in the paper, it was predicted that how the data are distributed should not affect the approximate bias.

This paper provides a general bias result. Instead of ARMA models of specific orders, bias approximation for a general ARMA(p, q) model with possible exogenous regressors is discussed. The error term in the model is not restricted to be normally distributed and the estimator considered is the QMLE that maximizes the sample likelihood function under normality. In the end, explicit analytical bias results are derived for the AR, MA, and error variance parameters. There are in total four main findings: (i) how the data are distributed has no effect on the bias results; (ii) the bias of the estimated parameters associated with the AR and MA terms depends only on themselves and the exogenous regressors; (iii) the QMLE of parameters associated with the exogenous regressors is approximately unbiased; (iv) the error variance is always estimated with a downward bias, whose magnitude is proportional to the number of exogenous regressors and the orders of AR and MA terms in the model. Note that in deriving the main results, this paper also gives the analytical asymptotic covariance matrix of the QMLE directly in terms of model parameters. The bias, as well as the asymptotic covariance matrix, can be readily calculated or programmed.

At first glance, this paper is closely related to Bao (2016). But it stands totally different and provides new contributions. Firstly, Bao (2016) was mainly to give guidance on how to calculate numerically the finite-sample bias of the QMLE in ARMA models by working out one by one the various matrices involved in the bias expansion, and not so much could be learned regarding the structure of the bias. Basically, a numerical algorithm was given in Bao (2016) that readers can follow to calculate the bias, but neither analytical results nor general conclusions regarding the bias structure were provided. This paper provides analytical results for different blocks of the whole parameter vector, pertaining to the exogenous regressors, the AR part, the MA part, and the error variance, respectively. Secondly, in Bao (2016) the effects of nonnormality in general remain unknown and it is predicted that the error skewness and kurtosis *may* contribute to the finite-sample bias of the QMLE, but in the ten specific models considered their effects disappear. This paper shows formally that the error skewness and kurtosis in fact do not matter for any of the bias blocks. Thirdly, Bao (2016) calculated the biases in ARMA(1,1) (with or without a constant term) and compared through simulations with the results using the bias formulae in Tanaka (1984) (and reported that Tanaka's (1984) formulae could give misleading results), but did not show how useful the bias results would be if they were used for the purpose of bias correction. In this paper, since the analytical bias results are available, a feasible bias-correction procedure follows directly, as well as one based on the indirect inference approach. Lastly, this paper includes exogenous regressors, whereas at most a constant term was included in Bao (2016). The introduction of exogenous regressors complicates the analysis substantially.

The plan of this paper is as follows. The next section contains the main results by providing two propositions, one relating to the asymptotic covariance matrix and the other one for the finite-sample bias. Section 3 discusses how to use the bias result for the purpose of bias correction and provides some Monte Carlo evidence. The last section concludes. The proofs are relegated to the Appendix, where some notation used in the propositions is introduced. Throughout, matrix dimensions are not explicitly mentioned, unless deemed necessary. $\boldsymbol{I}$, $\mathbf{1}$, $\mathbf{0}$, and $\mathbf{O}$ represent the identity matrix, vector of ones, zero vector, and zero matrix, respectively.

2. Main results

Consider the following ARMA(p, q) process

$$\left(y_t - \boldsymbol{x}_t'\boldsymbol{\beta}\right) - \sum_{i=1}^{p} \phi_i \left(y_{t-i} - \boldsymbol{x}_{t-i}'\boldsymbol{\beta}\right) = u_t - \sum_{j=1}^{q} \theta_j u_{t-j}, \tag{1}$$

where $\boldsymbol{x}_t$ is a $k \times 1$ vector of exogenous regressors which may contain a constant term, u_t is the error term with mean zero and variance σ^2. Collect the parameters as $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)'$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)'$, $\boldsymbol{\alpha} = (\boldsymbol{\beta}', \boldsymbol{\phi}', \boldsymbol{\theta}', \sigma^2)'$. The total number of parameters is $m = k + p + q + 1$. This ARMA specification arises naturally from a linear model with an ARMA-type error term:

$$y_t = \boldsymbol{x}_t'\boldsymbol{\beta} + \varepsilon_t,$$

where ε_t follows an ARMA (p, q) process $\varepsilon_t = \sum_{i=1}^{p}\phi_i\varepsilon_{t-i} + u_t - \sum_{j=1}^{q}\theta_i u_{t-j}$.

There are different ways to estimate $\boldsymbol{\alpha}$. The prominent one may be the maximum likelihood approach. The Gaussian sample log-likelihood function can be written as

$$\mathcal{L}(\boldsymbol{\alpha}) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{t=1}^{T} u_t^2, \tag{2}$$

[3] Note that the conditional QMLE was considered in Bao (2016) whereas Tanaka (1984) was based on the exact likelihood function. So the initial conditions may matter for the approximate bias. But Tanaka's (1984) bias expressions for the ARMA(1,1) model turned out not involving the initial conditions.