ARTICLE IN PRESS

Journal of Statistical Planning and Inference ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Embedding in law of discrete time ARMA processes in continuous time stationary processes

Argimiro Arratia [a], Alejandra Cabaña [b,*], Enrique M. Cabaña [c]

[a] *Universitat Politècnica de Catalunya, Spain*
[b] *Universitat Autònoma de Barcelona, Spain*
[c] *Universidad de la República, Montevideo, Uruguay*

A R T I C L E I N F O

Article history:
Received 15 January 2017
Received in revised form 28 September 2017
Accepted 4 January 2018
Available online xxxx

Keywords:
Discrete-time ARMA
Continuous-time ARMA
CARMA
Lévy process
Embedding

A B S T R A C T

Given any stationary time series $\{X_n : n \in \mathbf{Z}\}$ satisfying an ARMA(p, q) model for arbitrary p and q with infinitely divisible innovations, we construct a continuous time stationary process $\{x_t : t \in \mathbf{R}\}$ such that the distribution of $\{x_n : n \in \mathbf{Z}\}$, the process sampled at discrete time, coincides with the distribution of $\{X_n\}$. In particular the autocovariance function of $\{x_t\}$ interpolates that of $\{X_n\}$.

1. Introduction

The description of phenomena that evolve continuously in time is frequently performed by means of series of observations made at equally spaced time intervals. When some regularity is assumed on the conditions under which the observations are made, a discrete-time stationary series might be used as a model for the observations. It is worth noticing that such model can be applied either for equally spaced observations of a stationary phenomenon that evolves in continuous-time (for instance, the concentration of oxygen in blood of a newborn child measured at fixed intervals), or as the result of observing a periodic phenomenon at the same point of successive periods (for instance, the temperature at noon at a given location in the equator). In the first case, a continuous-time model for the stationary process should exist, associated to the discrete-time model which is embedded in it. This poses the problem of finding such a process. The following particular case of this problem has been extensively studied by several authors: Given the stationary process X_t, $t \in \mathbf{Z}$, that satisfies the discrete ARMA(p, q) model, denoted as DARMA(p, q), namely

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{k=0}^{q} \theta_k \epsilon_{t-k} \tag{1}$$

where ϵ_t is a white noise with finite variance, the problem of obtaining a stationary process x_t, $t \in \mathbf{R}$, satisfying a continuous version of the DARMA model (denoted CARMA(p, q)) such that when sampled at discrete times has the same autocovariance function as $\{X_t\}$, is known as *embedding* a discrete-time ARMA process in a continuous-time ARMA process.

* Corresponding author.
E-mail addresses: argimiro@cs.upc.edu (A. Arratia), acabana@mat.uab.ca (A. Cabaña), ecabana@fing.edu.uy (E.M. Cabaña).

https://doi.org/10.1016/j.jspi.2018.01.004
0378-3758/

Please cite this article in press as: Arratia A., et al., Embedding in law of discrete time ARMA processes in continuous time stationary processes. J. Statist. Plann. Inference (2018), https://doi.org/10.1016/j.jspi.2018.01.004.

Brockwell (1995, 2004) summarizes the construction of CARMA processes as follows: A CARMA(p, q) process, for $0 \leq q < p$, is defined formally as a stationary solution of a continuous analogue of (1), namely, the stochastic differential equation

$$a(\mathcal{D})Y_t = b(\mathcal{D})\mathcal{D}\Lambda_t, \quad t \in \mathbf{R} \tag{2}$$

where $\sigma > 0$ is a scale parameter, $\mathcal{D}$ denotes differentiation with respect to t, Λ is a second-order Lévy process, and

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$$
$$b(z) = b_0 + b_1 z + \cdots + b_q z^q$$

are polynomials of order p and q, respectively. These CARMA processes are linear functions of continuous vector autoregressive (CVAR) Markovian processes.

The embedding problem has been studied by several authors. The works by Chan and Tong (1987), He and Wang (1989), Brockwell (1995) and Brockwell and Brockwell (1999) established embeddings of some DARMA(p, q) processes in continuous ARMA(p, q), for $0 \leq q < p$. Huzii (2006) gave necessary and sufficient conditions for a DARMA process to be embedded in a CARMA process. Using the concept of generalized random process of Gel'fand and Vilenkin (1964), Brockwell and Hannig (2010) extended the above definition of CARMA processes to allow for $q \geq p$. However in this case the generalized CARMA process does not exist in the classical sense.

All these approaches to the embedding problem are only concerned with the covariance structure of the processes involved, not with their probability distributions besides the fact that, if the processes are Gaussian, the equality of the first- and second-order moments entails the equality of the probability laws. In general, the discretized version of the CARMA will not necessarily have the same law as the original DARMA.

We propose in this work a different approach to construct for any DARMA (p, q) a continuous stationary *embedding in law*. Our main result is the following:

Theorem 1. *Given the stationary causal DARMA(p, q) process X_t that satisfies (1) with infinitely divisible innovations ϵ_t, there exists at least one function $L : \mathbf{R}^+ \to \mathbf{R}$ decaying exponentially at infinity and a Lévy process Λ on $\mathbf{R}$, such that the stationary processes $x_t = \int_{-\infty}^{t} L(t-s)d\Lambda(s)$, $t \in \mathbf{R}$, have the same joint law as X_t, $t \in \mathbf{Z}$.*

The function L and the Lévy process Λ are not unique in general. Both must fulfil conditions related to the discrete noise ϵ (Eq. (12)).

Our result means that any stationary time series satisfying a DARMA(p, q) model for arbitrary p and q can be embedded in a continuous parameter stationary process. The construction makes use of a similar embedding for vectorial autoregressive (VAR) processes, driven by an infinitely divisible white noise.

The embedding x_t can be applied to analyze the properties of the paths on intervals between observed points, such that the maxima, minima or the properties of the sagittae, that is, the difference between x_t and the (non-stationary) linear interpolation joining the graph of observed values.

The rest of the paper is structured as follows: The following section contains a brief outline of the construction of the embedding. Section 3 describes the technical details of the general construction, and Section 4 discusses the selection of a functional parameter of the embeddings in order to get desirable probabilistic properties, based on the properties of the autocovariances of the embedding. Finally, Appendices A and B supply the detailed computation of the Jordan normal form appearing in Section 3.2.

2. Proposed scheme for the embedding

We consider a family of continuous parameter stationary processes

$$x_t = \int_{-\infty}^{t} L(t-s)d\Lambda_s$$

depending on a centred second-order Lévy process $\Lambda_s, s \in \mathbf{R}$, and on a square integrable function $L(t), t \in R^+$, and show that given any DARMA $X_t, t \in \mathbf{Z}$, the parameters Λ and L can be selected in order that $\{X_t : t \in \mathbf{Z}\}$ and $\{x_t : t \in \mathbf{Z}\}$ have the same law.

The construction of the embedding is performed in five steps:

(1) Express the scalar DARMA(p, q) as an r dimensional discrete vectorial autoregressive process DVAR(1) where $r = \max\{p, q+1\}$.
(2) Transform the DVAR(1) into a new vectorial process J-DVAR(1) associated to the Jordan canonical form J of the matrix defining the DVAR(1).
(3) Split the J-DVAR(1) into simpler processes associated to each of the Jordan blocks in J, of dimensions r_i, $\sum_i r_i = r$.

Please cite this article in press as: Arratia A., et al., Embedding in law of discrete time ARMA processes in continuous time stationary processes. J. Statist. Plann. Inference (2018), https://doi.org/10.1016/j.jspi.2018.01.004.