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Limit theory for explosive autoregression under conditional heteroskedasticity

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ABSTRACT

This paper studies an explosive autoregression with conditionally heteroskedastic innovations. The asymptotic distributions of LS, GLS, t-statistics, heteroskedasticity-consistent t-statistics and GLS t-statistics are derived for nonstationary local-to-unity and mildly explosive roots, in which ρ_n satisfies $n(1-\rho_n) \to [-\infty, 0)$. Combined with the existing results, we can cover the range of ρ_n such that $n(1-\rho_n) \to [-\infty, \infty]$. Some results on pure explosive cases are also discussed.

1. Introduction

This paper considers a limit theory of the explosive autoregression with conditionally heteroskedastic innovations. Guo and Phillips (2001) derived an asymptotic theory of LS and GLS estimator for the unit root ($\rho_n = 1$) autoregression under conditional heteroskedasticity (hereafter CHE). Recently, Andrews and Guggenberger (2012) studied asymptotic distributions of LS, GLS and heteroskedasticity-consistent (hereafter HC) t-statistics under CHE, covering stationary to unit root region such that $n(1-\rho_n) \to [0, \infty]$. With $\delta \in (0,1)$, $c > 0$, the corresponding parameter space includes fixed stationary ($|\rho| < 1$), mildly integrated ($\rho_n = 1 - \frac{c}{n^\delta}$), stationary local-to-unity ($\rho_n = 1 - \frac{c}{n}$) and unit root ($\rho_n = 1$). In this paper, we develop asymptotic distributions of LS, GLS, t-statistics, HC t-statistics and GLS t-statistics under CHE for nonstationary local-to-unity ($\rho_n = 1 + \frac{c}{n}$) and mildly explosive ($\rho_n = 1 + \frac{c}{n^\delta}$) roots. Under normally distributed innovations, we also study the pure explosive case ($\rho_n = 1 + c$).

For LS estimator of autoregressive parameter, the limit theory of Phillips (1987) and the Cauchy limit theory of Phillips and Magdalinos (2007) are still valid under some standard conditions on the CHE error process. The limit distribution of GLS estimator for nonstationary local-to-unity case is shown to be a mixture of two independent unit root type distributions, which parallels Guo and Phillips (2001) and Andrews and Guggenberger (2012). A new scaled Cauchy limit theory of GLS for mildly explosive case is developed. In all parameter regions, the explicit efficiency gain by GLS procedure can be evaluated by a common factor κ, which also appears in Guo and Phillips (2001; π_σ) and Andrews and Guggenberger (2012; $h_{2,7}$).

In addition, we explore the limit distributions of three studentized estimators: t-statistics and HC t-statistics, and GLS t-statistics. For nonstationary local-to-unity root, both statistics have the same limiting distributions as in Phillips (1987).

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Table 1
Summary of AR(1) limit theory under CHE.

Parameter regions	$r_n\left(\hat{\rho}_{n,LS}-\rho_n\right)$	$r_n\left(\hat{\rho}_{n,GLS}-\rho_n\right)$	$t_n\left(\rho_n\right)$	$T_n^*\left(\rho_n\right)$
$\lvert\rho\rvert<1$ or $\rho_n=1-\frac{c}{n^\delta}$	$N\left(0,V_{LS}\right)$	$N\left(0,V_{GLS}\right)$	$N\left(0,V_t\right)$	$N(0,1)$
$\rho_n=1-\frac{c}{n}$	$\frac{\int_0^1 I_c(r)dW_1(r)}{\int_0^1 I_c(r)^2dr}$	Mixture 2	$\frac{\int_0^1 I_c(r)dW(r)}{\left(\int_0^1 I_c(r)^2dr\right)^{1/2}}$	$\frac{\int_0^1 I_c(r)dW(r)}{\left(\int_0^1 I_c(r)^2dr\right)^{1/2}}$
$\rho_n=1$	$\frac{\int_0^1 W_1(r)dW_1(r)}{\int_0^1 W_1(r)^2dr}$	Mixture 1	$\frac{\int_0^1 W(r)dW(r)}{\left(\int_0^1 W(r)^2dr\right)^{1/2}}$	$\frac{\int_0^1 W(r)dW(r)}{\left(\int_0^1 W(r)^2dr\right)^{1/2}}$
$\rho_n=1+\frac{c}{n}$	$\frac{\int_0^1 I_c(r)dW_1(r)}{\int_0^1 I_c(r)^2dr}$	Mixture 2	$\frac{\int_0^1 I_c(r)dW(r)}{\left(\int_0^1 I_c(r)^2dr\right)^{1/2}}$	$\frac{\int_0^1 I_c(r)dW(r)}{\left(\int_0^1 I_c(r)^2dr\right)^{1/2}}$
$\rho_n=1+\frac{c}{n^\delta}$	C (Cauchy)	κC (newly scaled)	$N(0,1)$	$N(0,1)$
$\rho_n=1+c, u\sim N$	C (Cauchy)	?	$N(0,1)$	?

In mildly explosive case, we have a standard normal distribution for both statistics. It is interesting to have the common limit distribution for t-statistics and HC t-statistics under CHE, and this is a special feature of near unity asymptotics, unlike the stationary case. When we have a stochastically bounded $O_p(1)$ (but not degenerate) limit of squared sum of regressors ($\sum_{t=1}^n y_{t-1}^2$) under suitable normalizations, sometimes (not always though) we can take CHE effect out of limit distribution so both t-statistics and HC t-statistics have the same asymptotic distributions. This region is from stationary local-to-unity ($\rho_n=1-\frac{c}{n}$) to mildly explosive ($\rho_n=1+\frac{c}{n^\delta}$) region. However, they are different in stationary and pure explosive region. The asymptotics of HC t-statistics shows complete symmetry around unity (see the fifth column of Table 1). This symmetry may be useful for uniform inference of AR(1) model, for example, as in Andrews et al. (2011). In fact, the index set of Andrews et al. (2011, also see Andrews and Guggenberger (2014)) can be extended from $H=[0,\infty]$ to $H'=[-\infty,\infty]$. The limit theory of GLS t-statistics is also studied in these explosive cases, generalizing Theorem 6.1 of Guo and Phillips (2001).

Notation. We use standard notation. $\Longrightarrow$, $\longrightarrow^p$ and $\longrightarrow^{a.s.}$ represent convergence in distribution, convergence in probability and almost sure convergence, respectively. All limit theory assumes $n\to\infty$ so we oftentimes omit this condition. $\sim$ signifies "being distributed as" either exactly or asymptotically, depending on the contexts. $O(1)$ and $o(1)$ ($O_p(1)$ and $o_p(1)$) are (stochastically) asymptotically bounded or negligible quantities.

- W_1 and W_2 are independent standard Brownian motions.
- $I_c(r)=\int_0^r e^{c(r-s)}dW_1(s)$ and $J_c(r)=\left[E\left(\sigma_t^2\right)\right]^{1/2}I_c(r)=\int_0^r e^{c(r-s)}dB(s)$ are Ornstein–Uhlenbeck (O–U) processes satisfying $dJ_c(r)=cJ_c(r)dr+dB(r)$ and $J_c(r)=\left[E\left(\sigma_t^2\right)\right]^{1/2}I_c(r)$, where σ_t^2 is defined in Assumption A2 below.
- Given the AR(1) process $y_t=\rho_n y_{t-1}+u_t$, $\hat{\rho}_{n,LS}$ is the typical Least square estimator, while $\hat{\rho}_{n,GLS}=\left(\sum_{t=1}^n\left(\frac{y_{t-1}}{\sigma_t}\right)^2\right)^{-1}\sum_{t=1}^n\left(\frac{y_{t-1}}{\sigma_t}\right)\frac{y_t}{\sigma_t}$.
- $t_n\left(\rho_n\right)=\frac{\left(\hat{\rho}_{n,LS}-\rho_n\right)}{\hat{\sigma}}$ and $T_n^*\left(\rho_n\right)=\frac{\left(\hat{\rho}_{n,LS}-\rho_n\right)}{\hat{\sigma}_{HC}}$ where $\hat{\sigma}^2=\left(\frac{1}{n}\sum_{t=1}^n\hat{u}_t^2\right)\left(\sum_{t=1}^n y_{t-1}^2\right)^{-1}$, and $\hat{\sigma}_{HC}^2=\left(\sum_{t=1}^n y_{t-1}^2\right)^{-1}\sum_{t=1}^n y_{t-1}^2\hat{u}_t^2\left(\sum_{t=1}^n y_{t-1}^2\right)^{-1}$.

Quick summary. We can have an almost complete inventory of asymptotic results of AR(1) model under conditional heteroskedasticity, where the rate of convergence r_n, the statistics $\hat{\rho}_{n,i}$'s and their asymptotic variances V_i's, with $i=LS, GLS, t$ are different but will be clearly specified in the corresponding Sections below.

- Mixture 1 is $\kappa\left(\kappa\frac{\int_0^1 W_1(r)dW_1(r)}{\int_0^1 W_1(r)^2dr}+\sqrt{1-\kappa^2}\frac{\int_0^1 W_1(r)dW_2(r)}{\int_0^1 W_1(r)^2dr}\right)$ which was derived in Guo and Phillips (1998), and $\kappa=\sqrt{\frac{1}{E\left(\frac{1}{\sigma_t^2}\right)E\left(\sigma_t^2\right)}}<1$.
- Mixture 2 is $\kappa\left(\kappa\frac{\int_0^1 I_c(r)dW_1(r)}{\int_0^1 I_c(r)^2dr}+\sqrt{1-\kappa^2}\frac{\int_0^1 I_c(r)dW_2(r)}{\int_0^1 I_c(r)^2dr}\right)$.

As we can see from the above, LS limit theory is robust to CHE. Note that

$$\sqrt{\frac{V_{GLS}}{V_{LS}}}=\sqrt{\frac{E\left[y_{t-1}^2\right]^2}{E\left[y_{t-1}^2\sigma_t^2\right]E\left[\frac{y_{t-1}^2}{\sigma_t^2}\right]}}\propto\sqrt{\frac{1}{E\left(\frac{1}{\sigma_t^2}\right)E\left(\sigma_t^2\right)}}=\kappa,$$

hence the common factor κ represents the efficiency gain in all parameter region. Furthermore, $t_n\left(\rho_n\right)$ and $T_n^*\left(\rho_n\right)$ have the same limit theory in the near unity region, since we have the same standard normal limit theory for mildly explosive root.

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