ARTICLE IN PRESS

Journal of Statistical Planning and Inference ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Fiducial, confidence and objective Bayesian posterior distributions for a multidimensional parameter

Piero Veronese *, Eugenio Melilli

Bocconi University, Milano, Italy

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Confidence distribution
Jeffreys prior
Location-scale parameter model
Multinomial model
Natural exponential family
Reference prior

ABSTRACT

We propose a way to construct fiducial distributions for a multidimensional parameter using a step-by-step conditional procedure related to the inferential importance of the components of the parameter. For discrete models, in which the non-uniqueness of the fiducial distribution is well known, we propose to use the geometric mean of the "extreme cases" and show its good behavior with respect to the more traditional arithmetic mean. Connections with the generalized fiducial inference approach developed by Hannig and with confidence distributions are also analyzed. The suggested procedure strongly simplifies when the statistical model belongs to a subclass of the natural exponential family, called conditionally reducible, which includes the multinomial and the negative-multinomial models. Furthermore, because fiducial inference and objective Bayesian analysis are both attempts to derive distributions for an unknown parameter without any prior information, it is natural to discuss their relationships. In particular, the reference posteriors, which also depend on the importance ordering of the parameters, are the natural terms of comparison. We show that fiducial and reference posterior distributions coincide in the location-scale models, and we characterize the conditionally reducible natural exponential families for which the same coincidence holds. The discussion of some classical examples closes the paper.

1. Introduction

Fiducial distributions, after having been introduced by Fisher (1930, 1935) and widely discussed (and criticized) in the subsequent years, have been de facto brushed aside for a long time and only recently they have obtained new vitality. The original idea of Fisher was to construct a *distribution* for a parameter which includes all the information given by the data, without resorting to the Bayes theorem. This is obtained by transferring the randomness from the observed quantity given by the statistical model to the parameter. Originally Fisher considered a continuous sufficient statistic S with distribution function F_θ, depending on a real parameter θ. Let $q_\alpha(\theta)$ denote the quantile of order α of F_θ and let s be a realization of S. If $q_\alpha(\theta)$ is increasing in θ (i.e., F_θ is decreasing in θ), the statement $s < q_\alpha(\theta)$ is equivalent to $\theta > q_\alpha^{-1}(s)$ and thus Fisher assumes $q_\alpha^{-1}(s)$ as the quantile of order $1-\alpha$ of a distribution which he names *fiducial*. The set of all quantiles $q_\alpha^{-1}(s)$, $\alpha \in (0, 1)$, establishes the fiducial distribution function $H_s(\theta)$ so that

$$H_s(\theta) = 1 - F_\theta(s) \quad \text{and} \quad h_s(\theta) = \frac{\partial}{\partial\theta} H_s(\theta) = -\frac{\partial}{\partial\theta} F_\theta(s). \tag{1}$$

Of course H_s, and its density h_s, must be properly modified if F_θ is increasing in θ.

* Corresponding author.
E-mail address: piero.veronese@unibocconi.it (P. Veronese).

http://dx.doi.org/10.1016/j.jspi.2017.09.015

Fisher (1973, Ch.VI) also provides some examples of multivariate fiducial distributions obtained by a "step-by-step" procedure, but he never develops a general and rigorous theory. This fact, along with the problem to cover discrete models, the presence of some inconsistencies of the fiducial distribution (e.g. the marginalization paradox, see Dawid & Stone, 1982), and the difficulties in its interpretation, gave rise to a quite strong negative attitude towards Fisher proposal.

In the renewed interest for the fiducial approach a relevant role is played by the *generalized fiducial inference* introduced and developed by Hannig (2009, 2013), see also Hannig et al. (2016) for a review. He provides a formal and mathematically rigorous definition which has a quite general applicability. The crucial element of his definition is a data-generating equation $\mathbf{X} = \mathbf{G}(\mathbf{U}, \boldsymbol{\theta})$, which links the unknown parameter $\boldsymbol{\theta}$ and the observed data $\mathbf{X}$ through a random element $\mathbf{U}$ having a known distribution. Roughly speaking, by shifting the randomness of $\mathbf{U}$ from $\mathbf{X}$ to $\boldsymbol{\theta}$ (inverting $\mathbf{G}$ with respect to $\boldsymbol{\theta}$ after having fixed $\mathbf{X} = \mathbf{x}$), the distribution given by the statistical model leads to a distribution for the parameter $\boldsymbol{\theta}$. Contrary to the original idea of Fisher, the generalized fiducial distribution is non-unique and Hannig widely discusses this point. Applications to different statistical models can be found for instance in Hannig et al. (2007), Hannig and Iyer (2008) and Wandler and Hannig (2012).

Other recent contributions to the topic of fiducial distributions are given by Taraldsen and Lindqvist (2013), Martin and Liu (2013) and Veronese & Melilli (2015), henceforth V&M (2015). In this last paper the authors derive fiducial distributions for a parameter in a discrete or continuous real natural exponential family (NEF), and discuss some of their properties with particular emphasis on the frequentist coverage of the fiducial intervals.

In the past, fiducial distributions have often been associated with *confidence distributions* even if these latter have a different meaning. A modern definition of confidence distribution is given in Schweder and Hjort (2002) and Singh et al. (2005), see the book by Schweder and Hjort (2016) for a complete and updated review on confidence distributions and their connections with fiducial inference. It is important to emphasize that a confidence distribution must be regarded as a function of the data with reasonable properties from a purely frequentist point of view. A confidence distribution is conceptually similar to a point estimator: as there exist several unbiased estimators, several confidence distributions can be provided for the same parameter and choosing among them can be done resorting to further optimality criteria. Thus the confidence distribution theory allows to compare, in a quite general setting, formal distributions for the parameter derived by different statistical procedures.

In this paper we suggest a way to construct a unique distribution for a multidimensional parameter, indexing discrete and continuous models, following a step-by-step procedure similar to that used by Fisher (1973) in some examples. We call it *fiducial distribution*, but we look at it simply as a distribution on the parameter space in the spirit of the confidence distribution theory. The key-point of the construction is the procedure by conditioning: the distribution of the data is factorized as a product of one-dimensional laws and, for each of these, the fiducial density for a real parameter component, possibly conditional on other components, is obtained. The joint fiducial density for the parameter is then defined as the product of the (conditional) one-dimensional fiducial densities. It is well known that Fisher's fiducial argument presents several drawbacks in higher dimensions, essentially because one cannot recover the fiducial distribution for a function of the parameters starting from the joint fiducial distribution, see Schweder and Hjort (2016, Ch. 6 and 9). Our approach, when it can be applied, presents the advantage to construct sequentially the fiducial distribution directly on the parameters of interest and different fiducial distributions can be obtained focusing on different parameters of interest. Noticed that a general definition of confidence distribution for a multidimensional parameter does not exist and the attention is given to the construction of approximate confidence curves for specific nested families of regions, see Schweder and Hjort (2016, Ch. 9 and Sec. 15.4).

Interestingly, our joint fiducial distribution coincides in many cases with the Bayesian posterior obtained using the reference prior. This fact motivates the second goal of the paper: to investigate the relationships between the objective Bayesian posteriors and the suggested fiducial distributions. Objective Bayesian analysis, see e.g. Berger (2006), essentially studies how to perform a *good* Bayesian inference, especially for moderate sample size, when one is unwilling or unable to assess a subjective prior. Under this approach, the prior distribution is derived directly from the model and thus it is labeled as *objective*. The reference prior, introduced by Bernardo (1979) and developed by Berger and Bernardo (1992), is the most successful default prior proposed in the literature. For a multidimensional parameter the reference prior depends on the grouping and ordering of its components and, in general, no longer coincides with the Jeffreys prior. This latter coincides with the reference prior only for a real parameter and, as is well known, it is unsatisfactory for a multivariate parameter.

Lindley (1958) was the first to discuss the connections between fiducial and posterior distributions for a real parameter, when a real continuous sufficient statistic exists. V&M (2015) extend this result to real discrete NEFs, characterizing all families admitting a *fiducial prior*, i.e. a prior leading to a posterior coinciding with the fiducial distribution. This prior is strictly related to the Jeffreys prior. We show here that when the parameter is multidimensional this relationship no longer holds and a new connection is established with the reference prior. In particular we prove results for location-scale parameter models and *conditionally reducible* NEFs, a subclass of NEFs defined in Consonni and Veronese (2001).

The paper is structured as follows. Section 2 reviews some basic facts on fiducial and confidence distributions for real NEFs and on generalized fiducial distributions. The proposal for constructing a step-by-step multivariate fiducial distribution is presented in Section 3, which also discusses: the relationships with confidence distributions (Section 3.1), the use of the geometric mean of fiducial densities for solving the non-uniqueness problem in discrete models (Section 3.2), the connections with the generalized fiducial inference and the consistency with the sufficiency principle (Section 3.3). Section 3.4 studies the fiducial distributions for conditionally reducible NEFs and provides their explicit expression for a particular subclass which includes the multinomial and the negative-multinomial model. Section 4 analyzes the relationships between

Please cite this article in press as: Veronese P., Melilli E., Fiducial, confidence and objective Bayesian posterior distributions for a multidimensional parameter. J. Statist. Plann. Inference (2017), http://dx.doi.org/10.1016/j.jspi.2017.09.015.