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Technical Note

A modified Galerkin FEM for 1D Helmholtz equations

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ABSTRACT

A method is presented that aims to eliminate the numerical errors inherent in the standard Galerkin finite element method (GFEM) for solving homogeneous Helmholtz equations. An error analysis of the standard GFEM with linear elements is first performed by using the concept of truncation error in finite difference methods, and then the truncation error expression is obtained. A linear GFEM with an artificial stiffness is proposed to solve the Helmholtz equation after investigating the effect of the error on numerical solution. The proposed method is essentially as straightforward as the standard GFEM and thus requires almost no additional computational effort. Numerical results show that present pollution error decreases by 90% compared with that of the standard GFEM.

1. Introduction

The Helmholtz equation governing time-harmonic acoustic waves is fundamental in many physical applications such as underwater acoustics, duct acoustics, acoustic scattering analysis, and electromagnetic and elastic wave propagation. Solutions to the Helmholtz equation can be approximated by asymptotic methods if the wavelength is small enough relative to the characteristic scale of a problem [1]. However, if the wavelength is of the same order as the characteristic scale, the Helmholtz equation has to be solved by numerical methods, such as the prevailing finite element method (FEM) and boundary element method (BEM). The Galerkin FEM (GFEM) is one of the popular numerical methods for practical applications. The application of the GFEM to solve the Helmholtz equation requires the problem domain to be initially discretized into a large number of small elements, and then within each element, the fundamental variable is usually described in terms of simple polynomial shape functions. The differential Helmholtz equation finally becomes a set of algebraic equations after discretization.

It is well known that the quality of GFEM solutions strongly depends on both the wave number k and the element size, h. In order to ensure the accuracy of the GFEM solution, the element should be adjusted to an appropriate size corresponding to the wave number [1,2]. In practice, at least 6–10 linear elements per wavelength should be used [3] to ensure an appropriate numerical accuracy. This implies that a very small element size is required in the numerical model to obtain acceptable prediction accuracy of the solution in high frequencies, leading to a large system of algebraic equations for multi-dimensional large scale problems. Significant computer resources and computational effort are required, since the computational effort is proportional to the square of the number of algebraic equations. Therefore, the FE approximation becomes prohibitively expensive such that the practical use of the standard method is restricted to low-frequency applications. However, in many industrial applications, there is a strong demand for appropriate techniques capable of providing accurate vibro-acoustic analysis in the mid-frequency range. According to Refs. [4–7], an accurate prediction of the short wave field is regarded as one of the most challenging problems from a numerical simulation perspective.

Many enhancements and extensions of the standard GFEMs have been developed in the last decade aiming to improve the accuracy of simulation for short waves. Several extensive overviews of different methods have been conducted in [6–10]. They can be classified into three categories: high-order methods, stabilized methods and multiscale methods.

High-order methods involve increasing the order of the polynomials (shape functions) used, such as a p-refinement and an h-p combination [11]. The stabilized methods involve reducing the artificial dispersion error between numerical and physical wave numbers by introducing a mesh-dependent stability parameter and minimizing the pollution error [12–14]. Multiscale methods apply specific shape functions representing *a priori* information about the global, short-wavelength behavior in the local approximation

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field, instead of changing the integral formulation of the problem. These methods can be further divided into two groups: in the first group, such as in the case of the partition-of-unity FEM (PUFEM), the new basis function is the product of a free-space solution and a conventional polynomial basis function, and in the second group, the new basis function is the superposition of an analytical solution describing the global, short-wave (fine-scale solution) behavior and conventional polynomial basis functions (coarse-scale) [10,15,16].

In engineering computations, besides the requirements for high accuracy and reduced computational time, a numerical model is also required to be simple and convenient in application. It is well known from various existing GFEM models that the model described by a linear shape function is the simplest and most convenient to practically implement. However, its accuracy is usually unsatisfactory. Therefore, it is of great significance to improve the accuracy of numerical solution to the homogeneous Helmholtz equation.

In this paper, the concept of truncation error in finite difference methods is used first to analyze the numerical error of the standard GFEM for one dimensional homogeneous Helmholtz equation. Then an artificial stiffness (or anti-error term) is introduced to eliminate the error of the GFEM. Based on this treatment, a modified GFEM with a linear shape function is finally proposed. The present method is simple and straightforward to implement as is the existing standard GFEM.

2. Error analysis of the Galerkin FEM

The general form of homogeneous Helmholtz equation is

$$\nabla^2\Phi + k^2\Phi = 0 \tag{1}$$

where Φ is the acoustic pressure, ∇ is the gradient operator, and $k = \omega/c$ is the wave number where ω is the angular frequency and c the wave propagation speed.

It is well known that the error between the standard numerical solutions and the corresponding exact solution with a given boundary condition is unavoidable. This error consists of different components in which the interpolation and pollution errors are dominant. Traditionally, the error is estimated in terms of an average through the concept of a norm defined in an integrable function space, as in the work of Ihlenburg et al. [11], Bouillard et al. [17] and others [18–21]. Here, the error is analyzed from another perspective, i.e., through the difference between the Helmholtz equation and the algebraic equations resulting from the application of the GFEM at any interior node. In current paper, the investigation is restricted to the one-dimensional (1D) case. Eq. (1) for 1D case is reduced into

$$\frac{d^2\Phi}{dx^2} + k^2\Phi = 0,\ x \subset (0, l) \tag{2}$$

with suitable boundary conditions at both ends where l is the length of the problem domain.

Applying the standard GFEM procedure to Eq. (2) results in the following linear algebraic system of equations,

$$(\mathbf{K} - k^2\mathbf{M})\boldsymbol{\phi} = \mathbf{f} \tag{3}$$

where $\boldsymbol{\phi}$ is the vector consisting of acoustic pressures at all the nodes, and $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{f}$ are defined as follows,

$$\mathbf{K} = \int_\Omega \frac{d\mathbf{N}^T}{dx}\frac{d\mathbf{N}}{dx}dx$$
$$\mathbf{M} = \int_\Omega \mathbf{N}^T\mathbf{N}dx \tag{4}$$
and
$$\mathbf{f} = \left\{ -\frac{d\phi_1}{dx}\bigg|_{x=0},\quad 0,\quad \cdots,\quad 0,\quad \frac{d\phi_{n+1}}{dx}\bigg|_{x=l} \right\}^T$$

where $\mathbf{N}$ is the shape function consisting of a Lagrange polynomial of order p. An order $p = 1$ implies that a linear element (h-version) is used and an order $p = 2$ implies that a quadratic element (p-version) is used. The superscript T denotes the transpose of a matrix. The $\mathbf{K}$ and $\mathbf{M}$ matrices for single linear and quadratic elements are provided in Appendix A. It should be noted that for the convenience of later discussions, the symbol ϕ rather than Φ is used to represent the solution of (2).

Fig. 1 shows the exact solution (solid line), the linear interpolated solution (dotted line with unfilled square markers) and the GFEM solution (dash line with filled square markers) of the Helmholtz Eq. (2) with the following boundary conditions.

$$-\frac{d\phi}{dx}\bigg|_{x=0} + jk\phi|_{x=0} = 2jk \quad \text{and} \quad \frac{d\phi}{dx}\bigg|_{x=l} + jk\phi|_{x=l} = 0 \tag{5}$$

It can be seen that the error from the standard GFEM is large after the wave propagates half a wavelength and becomes even larger in the propagation direction. The error observed in Fig. 1, due to the phase difference between the standard GFEM and the exact solutions, is called the pollution error [4].

For a uniform mesh, substituting the linear shape function (h-version) into the algebraic Eq. (3) leads to a repeated finite element stencil centered at a typical interior node q, i.e.,

$$(a+b)\phi_{q-1} - 2(a-2b)\phi_q + (a+b)\phi_{q+1} = 0 \tag{6}$$

where $a = 1/h$, $b = k^2h/6$ and h is the element length.

Usually, the solution of (6) does not satisfy the differential Eq. (2) exactly. In fact, using a Taylor series expansion at node q, (6) becomes

$$\phi''_q + k^2\phi_q + \delta = 0 \tag{7}$$

where

$$\delta = \frac{(kh)^2}{6}\phi''_q + \left[\frac{6+(kh)^2}{3}\right]\left[\phi_q^{(4)}\frac{h^2}{4!} + \cdots + \phi_q^{(2m)}\frac{h^{2(m-1)}}{(2m)!} + \cdots\right] \tag{8}$$

is the truncation error as named in the content of the finite difference method, and $\phi_q^{(2m)}$ denotes the $2m$th-order derivative of function ϕ_q.

Clearly, the truncation error, δ, which is caused by the interpolation and discretization, changes with the mesh size, h, of the FEM model. The quality of FEM solutions will depend on the magnitude of the truncation error δ: the smaller the value of δ, the higher the accuracy of the numerical solution. In addition, it is difficult to calculate the value of δ exactly, as the numerical solution $\{\phi_q\}$ is merely a set of function values at nodal points and does not contain higher order derivatives. However, expression (8) can be simplified by means of the characteristic of the Helmholtz equation, which states that any higher order derivatives can be transformed into

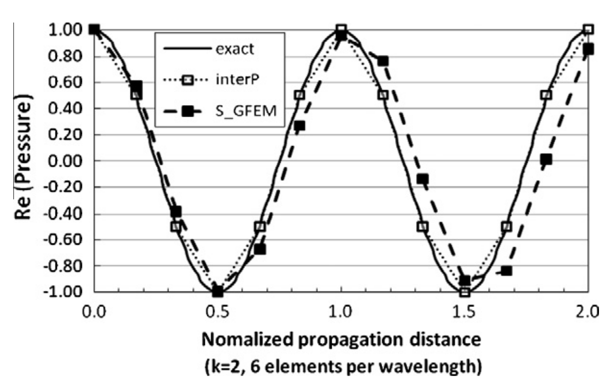

Fig. 1. Pressure difference between the exact and standard GFEM solutions.