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Distribution-free tests of conditional moment inequalities[☆]

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ABSTRACT

This article proposes testing the hypothesis of a uniformly non-positive nonparametric regression function using a test statistic with tabulated critical values. The null hypothesis is characterized in terms of the significance of a parameter, which measures a distance from the double-integrated regression function to the class of concave functions. The test statistic is a suitably scaled parameter estimate, which does not require smooth estimation of the underlying regression and/or the conditional variance functions. The finite sample performance of the proposed test is studied by means of two Monte Carlo experiments, showing that the proposed method compares favorably to existing procedures.

1. Introduction and summary

Let (Y, X) be a bivariate random vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Assume Y is integrable so that the regression function $m(X) := \mathbb{E}(Y|X)$ is well defined almost surely (*a.s.*). This article proposes testing the hypothesis

$$H_0 : m(X) \leq 0 \quad a.s., \tag{1}$$

in the direction of non-parametric alternatives H_1, which consists of all cases where H_0 is not satisfied.

Inequality restrictions such as (1) appear naturally when testing treatment effects controlling for covariates. Let D be an indicator of participation in a treatment program, i.e. $D = 1$ if the individual participates in the program and 0 otherwise. Denote the observed outcome by $Z = Z(1)D+Z(0)(1-D)$, where $Z(1)$ and $Z(0)$ are the potential outcomes with and without treatment, respectively. The treatment is successful uniformly in the covariate X, e.g. age, if $\mathbb{E}(Z(0) - Z(1)|X) \leq 0$ *a.s.*, which can be expressed as (1) with $Y = (\mathbb{E}(D|X) - D)Z$, provided $0 < \mathbb{E}(D|X) < 1$ *a.s.* and the treatment is randomized conditional on covariates, i.e. $Z(1)$ and $Z(0)$ are independent of D, conditional on X. See Delgado and Escanciano (2013) and Chang et al. (2015) for further discussion. Identifiability conditions on econometric models often appear as testable restrictions on moment inequalities. For instance, behavioral choice models generate conditional moment inequalities suitable to identifying parameters of nonparametric functions of interests; see Pakes (2010). This includes testing the "realistic expectation hypothesis" in insurance market modeling; e.g. Chiappori et al. (2006). Inference on game theoretical models often assume that some underlying regression function is non-negative; see de Paula (2013) for a survey. Inequality restrictions on conditional models also arise when testing revealed preferences; see e.g. Blundell et al. (2003). Finally, partial

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identification conditions can often be written as conditional moment inequalities. Some references on inference procedures on moment inequalities are Khan and Tamer (2009), Chernozhukov et al. (2013), Armstrong (2015) and references therein.

Under (1), $\mathbb{E}\left(Y \cdot 1_{\{a \le X \le b\}}\right) \le 0$ for all $a, b \in \mathbb{R}$. This fact has suggested tests of (1) based on local averages. Dümbgen and Spokoiny (2001) and Juditsky and Nemirovski (2002) proposed a test of qualitative hypotheses on the signal of a Gaussian white noise model, which include positivity, based on kernel estimators of the regression function. The resulting test is adaptive in the class of smooth functions considered. Baraud et al. (2003, 2005) proposed a test based on trimmed averages for qualitative hypotheses on the regression function of a fixed design model with homoskedastic errors. Exact critical values of these tests are derived under Gaussian errors. Asymptotic tests of positivity in the context of general models with random covariates and possibly non Gaussian errors have been recently proposed by Kim (2008), Andrews and Shi (2013), Chetverikov (2013) and Armstrong (2015), among others. The critical values of these tests must be estimated with the assistance of bootstrap techniques. The test of Lee et al. (2013) is of a different nature. The test statistic is based on a one-sided version of the L^p-*type* functionals of kernel estimators using standard normal critical values. The asymptotic test is justified when the bandwidth converges to zero at a suitable rate related to the sample size and assuming different restrictions on m. See also Lee et al. (2014). The bootstrap test of Delgado and Escanciano (2013), related to Durot (2003) and Delgado and Escanciano (2011) monotonicity tests, avoids estimating the regression function. In the general case, the limiting distribution of their test depends on a nuisance parameter, the integrated conditional variance. This article applies this testing methodology to construct a test with pivotal critical values, free of nuisance and tuning parameters.

Henceforth, let F denote the cumulative distribution function (cdf) of X, which is assumed to be continuous, and for a generic monotone function $G : \mathbb{R} \to \mathbb{R}$, let G^{-1} denote its generalized inverse $G^{-1}(r) := \inf\{t \in \mathbb{R} : G(t) \ge r\},\ r \in \mathbb{R}$. The null hypothesis can be equivalently expressed as

$$H_0 : M \text{ is non-increasing},$$

where

$$M(u) = \mathbb{E}\left[Y \cdot 1_{\{F(X) \le u\}}\right] = \int_0^u \left(m \circ F^{-1}\right)(v)dv, \quad u \in [0, 1]$$

is the integrated regression function, and $\circ$ denotes composition of functions. This, in turn, is satisfied if

$$H_0 : \mathbb{M} \text{ is concave},$$

where

$$\mathbb{M}(u) := \int_0^u M(v)dv, \quad u \in [0, 1].$$

We exploit this fact, expressing H_0 as a significance test on a parameter by using the least concave majorant (lcm) operator $\mathcal{L}$, defined as follows. For any function $g : [0, 1] \to \mathbb{R}$, (i) $\mathcal{L}g$ is concave and (ii) if there exists a concave function h with $h \ge g$, then $h \ge \mathcal{L}g$. Let $\|\cdot\|$ be a norm defined on the space of continuous functions satisfying the Riesz's property, i.e. if $0 \le g(u) \le h(u)$, for all $u \in [0, 1]$, then $\|g\| \le \|h\|$. Examples of possible norms $\|\cdot\|$ include the sup-norm $\|g\|_\infty := \sup_{u\in[0,1]} |g(u)|$ and the L_2-norm $\|g\|_2^2 := \int_0^1 g^2(u)du$. The hypotheses can be alternatively expressed in terms of the parameter $\eta = \|\mathcal{L}\mathbb{M} - \mathbb{M}\| \ge 0$, i.e.

$$H_0 : \eta = 0 \quad \text{vs.} \quad H_1 : \eta > 0.$$

The parameter η measures a distance from m to the class of non-negative functions.

Given a random sample $\{(Y_i, X_i)\}_{i=1}^n$ of independent and identically distributed (i.i.d.) copies of (Y, X), the test statistic is based on an estimator of η. First, the integrated regression function $M(u)$ is estimated by,

$$\hat{M}(u) = \frac{1}{n}\sum_{i=1}^n Y_i \cdot 1_{\left\{\hat{F}(X_i) \le u\right\}},$$

where $\hat{F}(\cdot) := n^{-1}\sum_{i=1}^n 1_{\{X_i \le \cdot\}}$ is the empirical analog of F. Henceforth, we do not indicate the dependence of the statistics on the sample size n. This suggests the use of the following estimator of η

$$\hat{\eta} = \left\|\mathcal{L}\widehat{\mathbb{M}} - \widehat{\mathbb{M}}\right\|,$$

with

$$\widehat{\mathbb{M}}(u) = \int_0^u \hat{M}(v)dv = \frac{1}{n}\sum_{i=1}^n Y_i \cdot \left[u - \hat{F}(X_i)\right] 1_{\left\{\hat{F}(X_i) \le u\right\}}.$$

Since $\hat{\eta}$ is expected to take small values under H_0 and large values under H_1, a scaled version of $\hat{\eta}$ could be used as a test statistic. A related testing strategy was suggested by Durot (2003) in order to test that m is monotonic in the context of a fixed design model with homoskedastic errors. In the general case, asymptotic critical values for tests based on $\hat{\eta}$ depend on the integrated variance $\tau(u) := \int_0^u \left(\sigma^2 \circ F^{-1}\right)(v)dv$, where $\sigma^2(\cdot) := \mathrm{Var}\left(Y | X = \cdot\right)$. Delgado and Escanciano (2013) suggested a bootstrap test using $\hat{\eta}$ as test statistic. In contrast, this article proposes a modification of $\hat{\eta}$, so that the resulting test is asymptotically pivotal. The main contributions of the article are summarized as follows: