Journal of Statistical Planning and Inference 171 (2016) 130–134

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Short communication

On the time for Brownian motion to visit every point on a circle

Philip Ernst [*], Larry Shepp [1]

Department of Statistics, Rice University, Houston, TX 77005, USA

ARTICLE INFO

Article history:
Received 15 June 2015
Received in revised form 20 October 2015
Accepted 21 October 2015
Available online 30 October 2015

MSC:
primary 60J65
secondary 60G15

Keywords:
Range of Wiener process
Continuous recurrence
First hitting time

ABSTRACT

Consider a Wiener process W on a circle of circumference L. We prove the rather surprising result that the Laplace transform of the distribution of the first time, θ_L, when the Wiener process has visited every point of the circle can be solved in closed form using a continuous recurrence approach.

1. Introduction

Consider a Wiener process on a circle of circumference L. The distribution of the first time, θ_L, when the Wiener process has visited every point of the circle is equivalent, via the natural bijection between and interval of the form $[b, b+L)$ on the real line and a circle of circumference L, to the distribution of the first time when the range of the Wiener process on the real line is of length L. This distribution is well-known and it has the following Laplace transform: [see, for example, (Borodin and Salminen, 2002), p. 242]

$$\mathbb{E}\left[e^{-s\theta_L}\right] = \frac{1}{\cosh^2\left(L\sqrt{\frac{s}{2}}\right)}, \quad s \geq 0. \tag{1}$$

Feller (1951), in writing about the range of a Wiener process, did so using explicit probability density calculations. Imhof (1986) discovered Laplace transform for the first time, θ_L, when the Wiener process has visited every point of the circle, again via explicit probability density calculations. Further computations employing the Laplace transform for θ_L were presented in Vallois (1993). However, in departure from these previous works, we prove the result in Eq. (1) using a continuous recurrence setup. We do so by calculating the left hand side in terms of random variables representing how long it takes to cover a range of length L, *given that one is already at an endpoint of a range of length a* (which counts as being covered already). This is the idea behind the definition of $\theta_{a,L}$, which is defined in Section 2.

* Corresponding author.
E-mail address: philip.ernst@rice.edu (P. Ernst).

[1] Deceased April 23, 2013.

http://dx.doi.org/10.1016/j.jspi.2015.10.010

Key to our recurrence will be the concept of a *switchback*. Imagine we pick some $a \in \mathbb{R}^+$ that is less than L. Consider the maximum, M_a, of W until the first visit to the point, $-a$ on the negative half-axis. (Here, $M_a > 0$; otherwise, the process would have moved directly from 0 to $-a$, which occurs w.p. 0) We call the time of this first visit τ_{-a}. We say that a "*switchback*" occurs when W hits $-a$ before the length of the range, $a + M_a$, is L. Formally, let $\mathbb{1}_{a,L}$ be the indicator random variable for the event of a switchback, defined as follows:

$$\mathbb{1}_{a,L} = \begin{cases} 1 & \text{if } \inf\{t : 0 \le t < \infty \mid W_t = -a\} \le \inf\{t : 0 \le t < \infty \mid W_t = L - a\} \\ 0 & \textit{otherwise}. \end{cases}$$

After a switchback, the process continues from $-a$ with a starting range of $M_a + a$ (i.e., the interval $[-a, M_a]$ has been covered). By translation and reflection invariance, as well as the symmetry of Brownian motion, we may just as well assume that we are at the point 0 and have covered the interval $[-(a + M_a), 0]$. We then repeat the process and say that a second switchback occurs if we reach $-(a + M_a)$ before covering a range of length L. To summarize:

Step 1: We start our process at the right hand end of $[-a, 0]$ and we consider this interval as already being covered. M_a is the maximal value attained before the time τ_a that we first hit $-a$. The total range is $a + M_a$. If $M_a \ge L - a$, then we have covered an interval of length L before reaching $-a$, and no switchback occurs. If not, a switchback occurs and we continue to Step 2.

Step 2: We have covered a range of length $a+M_a$. Without loss of generality, we consider the interval $[-(a+M_a), 0]$ to have been covered. Let $-(a+M_a) := -a'$, and start the process on the right hand end of $[-a', 0]$. If $M_{a'} \ge L - M_{a'}$, no switchback occurs. Otherwise, another switchback occurs and we continue to Step 3.

Step 3: We have covered a range of length $a' + M_{a'}$. Without loss of generality, we consider the interval $[-(a' + M_{a'}), 0]$ to have been covered. Let $-(a' + M_{a'})$ be called $-a''$, and start the process on the right hand end of $[-a'', 0]$. If $M_{a''} \ge L - a''$, a switchback occurs. Otherwise, continue Step 3 recursively until a range of length L has been covered.

Steps 1–3 are illustrated in Fig. 1.

In Section 3 we prove that the recurrence can be solved in closed form. In Section 4 we prove that the number $\nu = \nu_{a,L}$, of switchbacks before covering an interval of length L has a Poisson distribution with parameter $\lambda = \log \frac{L}{a}$. Thus, as $a \downarrow 0$, the number of switchbacks goes to infinity at a logarithmic rate.

2. Solving the recurrence

We proceed to solve for the recurrence. First, consider a Wiener process $W(t),\ t \ge 0$. For each fixed, $a > 0$, let M_a denote the maximum positive value of $W(t)$ before the first hitting time of $-a$. Assuming that $L - a$ is positive, we have

$$\mathbb{P}(M_a \le y) = \mathbb{P}\left(\tau_{-a} < \tau_y\right) = \frac{y}{a+y},$$

by the logic of the gambler's ruin.

Let $I(t)$ be the range of the Wiener process up to time t. Define $\theta_{a,L}$ to be the random variable representing the time until $I(t) \cup [-a, 0]$ has length L. We proceed by defining

$$f(s, a, L) := \mathbb{E}\left[\exp\left(-s\theta_{a,L}\right)\right], \tag{2}$$

where $f(s, a, L)$ is considered a function of a with s and L being held constant. By abuse of notation, we label $f(s, a, L)$ as $f(a)$.

Let us define the following functions

$$F(s, y) = \mathbb{E}\left[\exp(-s\tau_{-a})\, \mathbb{1}_{\tau_{-a}<\tau_y}\right] \quad \text{and} \quad G(s, y) = \mathbb{E}\left[\exp\left(-s\tau_y\right) \mathbb{1}_{\tau_y<\tau_{-a}}\right]. \tag{3}$$

We now employ the well-known fact (see Borodin and Salminen, 2002, amongst other sources), that for any c,

$$\exp\left(cW(t) - \frac{c^2}{2}t\right) \quad t \ge 0 \tag{4}$$

is a martingale. If $s = \frac{c^2}{2}$, we easily obtain the following standard and well known forms of $F(s, y)$ and $G(s, y)$ (see Borodin and Salminen, 2002, amongst other sources),

$$F(s, y) = \frac{\sinh cy}{\sinh(c(a+y))} \quad \text{and} \quad G(s, y) = \frac{\sinh ca}{\sinh(c(a+y))}. \tag{5}$$

Continuing from above, our goal is to write a recurrence for $f(a)$ in terms of $f(a + y)$ for $0 < y \le L - a$. To do so, we define $f(a)$ using indicator functions. With the process starting at 0, let the first indicator function represent the case of a switchback, in which $-a$ is hit before the length of the range is L. Let the second indicator function denote the case of no switchback. We may then write

$$f(a) = \underbrace{\mathbb{E}\left[\exp\left(-s\theta_{a,L}\right) \mathbb{1}_{\tau_{-a}<\tau_{L-a}}\right]}_{\text{switchback}} + \underbrace{\mathbb{E}\left[\exp\left(-s\theta_{a,L}\right) \mathbb{1}_{\tau_{L-a}<\tau_{-a}}\right]}_{\text{no switchback}}. \tag{6}$$