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On the construction of asymmetric orthogonal arrays

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ABSTRACT

A general method of construction of asymmetric orthogonal arrays was proposed by Suen et al. (2001). Using this method and some modifications thereof, we construct some more families of asymmetric orthogonal arrays of strength greater than two.

1. Introduction and preliminaries

Asymmetric orthogonal arrays introduced by Rao (1973) have received considerable attention in recent years. Such arrays are useful in experimental designs as universally optimal fractions of asymmetric factorials. Asymmetric orthogonal arrays have also been found to be very useful in industrial experimentation for quality improvement. Construction of asymmetric orthogonal arrays of strength *two* has been an area of intense research and we refer the reader to Hedayat et al. (1999) for an excellent description of these; see also Wu et al. (1992) and Chen et al. (2014). Relatively less is known on the construction of asymmetric orthogonal arrays of strength larger than two. Apart from the methods of construction of asymmetric orthogonal arrays of strength larger than two described in Dey and Mukerjee (1999) and Hedayat et al. (1999), further work on the construction of arrays of strength three or higher have been carried out e.g., by Suen et al. (2001), Suen and Dey (2003), Nguyen (2008) and Jiang and Yin (2013). In particular, Suen et al. (2001) proposed a general method for constructing asymmetric orthogonal arrays of arbitrary strength. This method was then applied by them to obtain several families of asymmetric orthogonal arrays of strength three and four. Suen and Dey (2003) combined tools from finite projective geometry with the method of Suen et al. (2001) to construct some new families of asymmetric orthogonal arrays of strength three and four.

Recall that an orthogonal array $OA(N, u, s_1 \times \cdots \times s_u, g)$ of strength g, is an $N \times u$ matrix with symbols in the ith column from a finite set of $s_i (\geqslant 2)$ symbols, $1 \leqslant i \leqslant u$, such that in every $N \times g$ submatrix, all possible combinations of symbols appear equally often as a row. Orthogonal arrays with $s_1 = s_2 = \cdots = s_u = s$ (say) are called symmetric and are denoted by $OA(N, u, s, g)$; otherwise, the array is called *asymmetric* (or, mixed levels). It is known that for an orthogonal

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array $OA(N, u, s_1 \times \cdots \times s_u, 3)$,

$$N \geqslant 1 + \sum_{i=1}^{u}(s_i - 1) + (s^* - 1)\left\{\sum_{i=1}^{u}(s_i - 1) - (s^* - 1)\right\},$$

where $s^* = max_{1 \leqslant i \leqslant u} s_i$. Arrays of strength three for which N attains the above bound are called *tight*.

Henceforth, the columns of an $OA(N, u, s_1 \times \cdots \times s_u, g)$ will be called *factors*, following the terminology in factorial experiments, and these factors will be denoted by $F_1, \ldots, F_u$. Throughout this paper, we take the integer $s \geqslant 2$ to be a prime or a prime power, i.e., $s = m^k$, where m is a prime and $k \geqslant 1$ is an integer. The Galois field of order s will be denoted by $GF(s)$, 0 and 1 being the identity elements of the field corresponding to the operations 'addition' and 'multiplication', respectively. Also, throughout a prime will denote transposition. We shall need the following results, the first of which is well known and the second one is due to Suen et al. (2001).

Lemma 1. *Let α and β be two elements of $GF(s)$ such that $\alpha^2 = \beta^2$. Then* (i) *$\alpha = \beta$ if s is even,* (ii) *either $\alpha = \beta$ or $\alpha = -\beta$, if s is odd.*

Lemma 2. *For a positive integer p, let D be a $(2p + 1) \times s^p$ matrix with columns of the form $(\alpha_1^2, \ldots, \alpha_p^2, \alpha_1, \ldots, \alpha_p, 1)'$, where $(\alpha_1, \ldots, \alpha_p)$'s are all possible p— tuples with entries from $GF(s)$. Then any three distinct columns of D are linearly independent.*

If $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are the elements of $GF(s)$, then it follows from Lemma 1 that the set $S = \{\alpha_0^2, \alpha_1^2, \ldots, \alpha_{s-1}^2\}$ contains all the elements of $GF(s)$ if s is even. If s is odd, then one element of S is 0 and there are $(s - 1)/2$ distinct non-zero elements of $GF(s)$, each appearing twice in S.

For the factor F_i $(1 \leqslant i \leqslant u)$, define the $r \times 1$ columns, $\boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{in_i}$ with elements from $GF(s)$. Then, for the u factors we have in all $\sum_{i=1}^{u} n_i$ columns. Also, let B be an $s^r \times r$ matrix whose rows are all possible r-tuples over $GF(s)$. Suen et al. (2001) proved the following result.

Theorem 1. *Consider an $r \times \sum_{i=1}^{u} n_i$ matrix $C = [P_1 \vdots P_2 \vdots \cdots \vdots P_u]$, $P_i = [\boldsymbol{p}_{i1}, \ldots, \boldsymbol{p}_{in_i}]$, $1 \leqslant i \leqslant u$, such that for every choice of g matrices $P_{i_1}, \ldots, P_{i_g}$ from $P_1, \ldots, P_u$, the $r \times \sum_{j=1}^{g} n_{i_j}$ matrix $[P_{i_1}, \ldots, P_{i_g}]$ has full column rank over $GF(s)$. Then an $OA(s^r, u, (s^{n_1}) \times (s^{n_2}) \times \cdots \times (s^{n_u}), g)$ can be constructed.*

A little elaboration of the result in Theorem 1 seems to be in order to make the construction transparent. For a fixed choice of g indices $\{i_1, \ldots, i_g\} \in \{1, \ldots, u\}$, let $C_1 = [P_{i_1}, \ldots, P_{i_g}]$ and $d = \sum_{j=1}^{g} n_{i_j}$. By the rank condition of Theorem 1, it follows that in the product BC_1, each possible $1 \times d$ vector with entries from $GF(s)$ appears s^{r-d} times. Now, for each j, $1 \leqslant j \leqslant g$, replace the $s^{n_{i_j}}$ distinct combinations under P_{i_j} by $s^{n_{i_j}}$ distinct symbols using a 1–1 correspondence. In the resultant $s^r \times g$ matrix,

(i) the i_jth column has $s^{n_{i_j}}$ symbols $(1 \leqslant j \leqslant g)$ and (ii) each of the $\prod_{j=1}^{g} s^{n_{i_j}}$ combinations of the symbols occurs equally often as a row. Hence, the desired orthogonal array with parameters as in Theorem 1 can be constructed.

This paper is organized as follows. Two families of asymmetric orthogonal arrays of strength three are constructed in Section 2. Using a replacement procedure, two new families of orthogonal arrays of strength $g \geq 2$ are reported in Section 3. In Section 4, we construct some families of *tight* asymmetric orthogonal arrays of strength three.

2. Construction of orthogonal arrays of strength three

In this section, we construct two families of orthogonal arrays of strength three.

Theorem 2. *Let $q \geqslant 2$ be an integer and l denote the largest integer not exceeding $q/2$.*

(i) *If s is an odd prime or an odd prime power, then an orthogonal array $OA(s^{2q+1}, s^q + (q - 1)(s - 1)^l + 2, (s^2) \times s^{s^q + (q-1)(s-1)^l + 1}, 3)$ can be constructed.*

(ii) *If s is a power of two, then an orthogonal array $OA(s^{2q+1}, s^q + (q - 1)s^l + 2, (s^2) \times s^{s^q + (q-1)s^l + 1}, 3)$ can be constructed.*

Proof. (i) Let s be an odd prime or an odd prime power. Let F_1 have s^2 symbols and the rest of the factors have s symbols each. The matrices P_i, $1 \leqslant i \leqslant u$, corresponding to the different factors are chosen as below, where $u = s^q + (q - 1)(s - 1)^l + 2$.

P_1 is chosen as $P_1 = \begin{bmatrix} I_2 & \mathbf{0}_{2,2q-2} & \boldsymbol{e} \end{bmatrix}'$,where I_a is the identity matrix of order a, $\mathbf{0}_{a,b}$ is a $a \times b$ null matrix, $\boldsymbol{e} = (0, 1)'$. The matrix P_2 is chosen as $A_2 = [0, x, \mathbf{0}_{1,2q-2}, 1]'$, $x \in GF(s)$, $x \neq 0, 1$.

Suppose $\gamma_1, \ldots, \gamma_l$ are non-zero elements of $GF(s)$. For $3 \leqslant i \leqslant (s - 1)^l + 2$, if q is even, then P_i is chosen to be of the form $P_i = [\gamma_1^2, \ldots, \gamma_l^2, \gamma_l, \ldots, \gamma_1, 0, 1, \mathbf{0}_{1,q-1}]'$. For $(s - 1)^l + 3 \leqslant i \leqslant 2(s - 1)^l + 2$, let P_i be of the form

$$P_i = [\gamma_1^2, \ldots, \gamma_l^2, \gamma_l, \ldots, \gamma_1, 0, 0, 1, \mathbf{0}_{1,q-2}]',$$

i.e., 1 appears in the $(q + 3)$th position. The other A_i matrices for this case are obtained by putting 1 in the $(q + 4)$th, $(q + 5)$th, ..., $2q$th position, to get a total of $(q - 1)(s - 1)^l$ columns of such a form. If q is odd, P_i is of the form